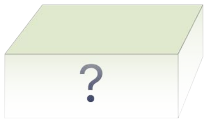
Thanks!

Dr. Alan Jones

Fundamentals of Inversion

Doug Oldenburg, Seogi Kang, Lindsey Heagy
& the UBC-GIF team



Collaborators

Seogi

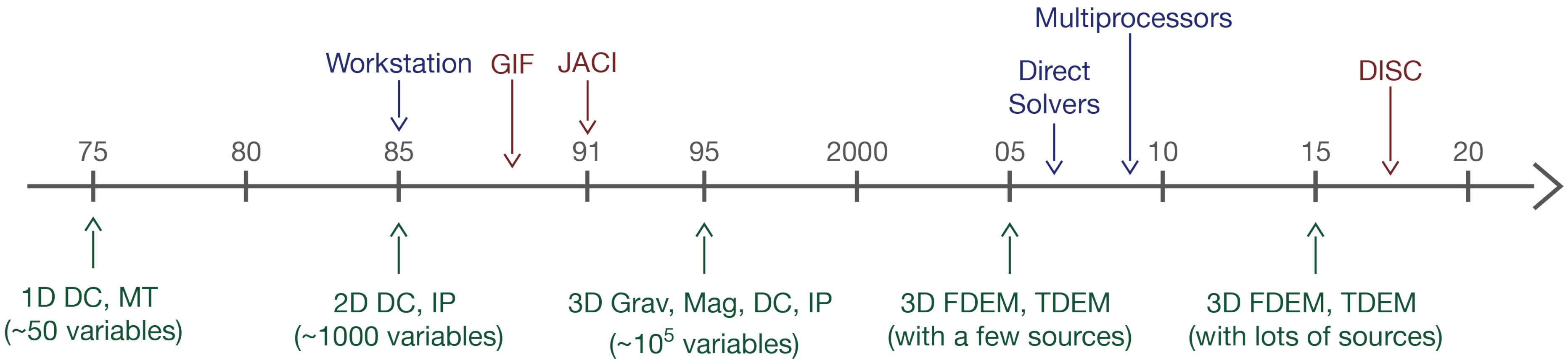
Lindsey

Dom

Thibaut

Some background and personal perspective

- Doug inspired by Bob Parker, Freeman Gilbert and George Backus: The Geophysical Inverse Problem



Result: Computing power + advances in inversion methodology → we can now solve most EM geophysics problems

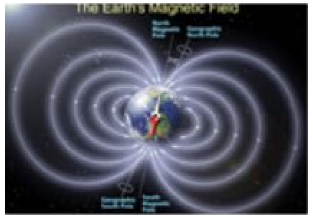
Outline

- Choices for numerical implementation
- Linear Inverse problem (IP)
- Non-linear inverse problem (DC)
- Including other information
- Summary

Generic geophysical experiment?

All require ways to see into the earth without direct sampling

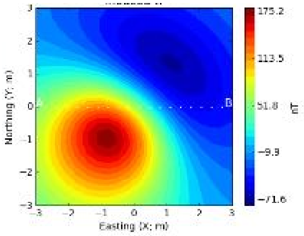
Source



Input energy

measure

Data

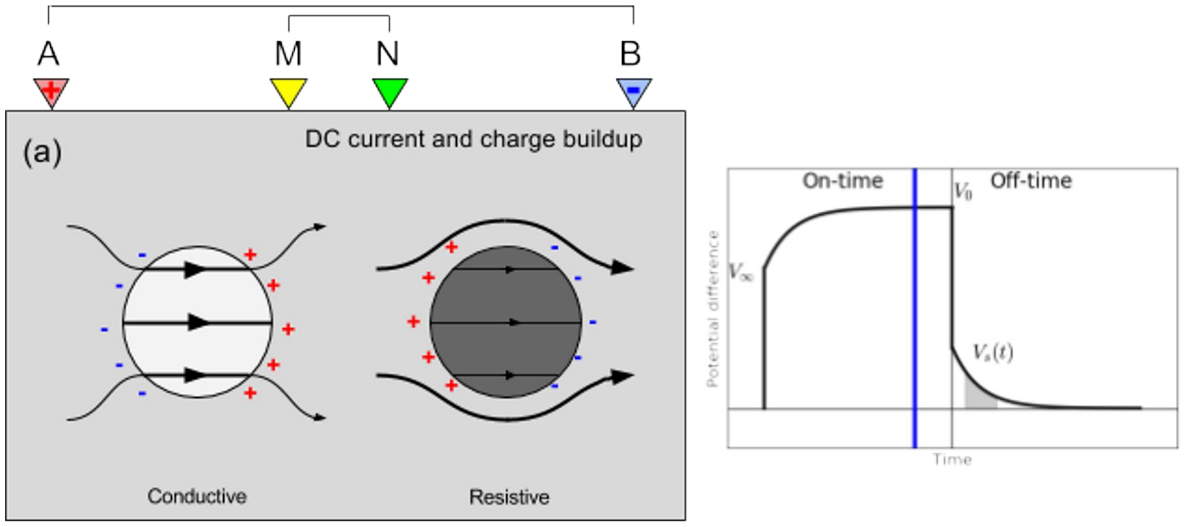


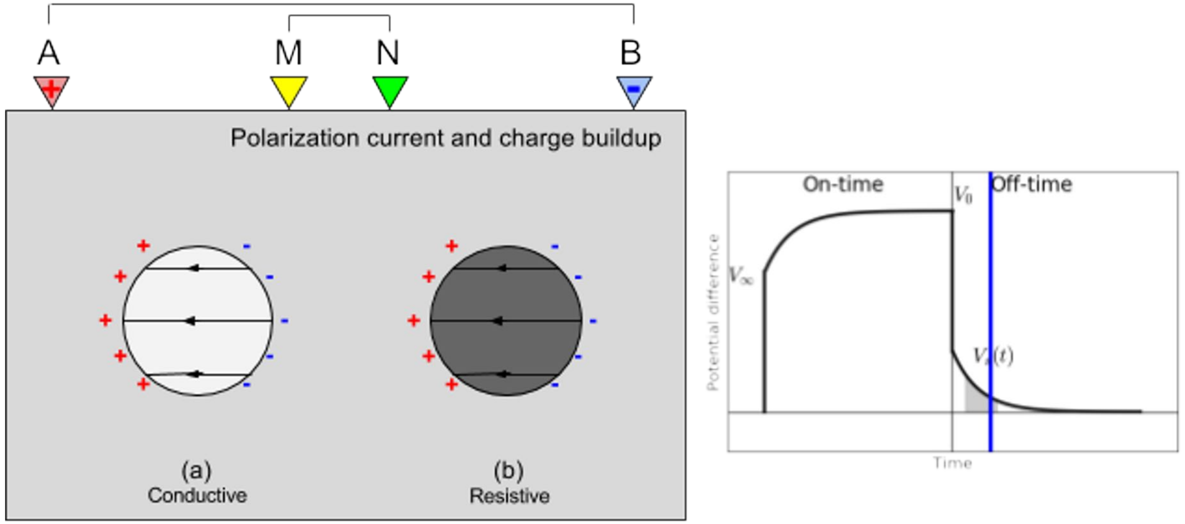
Physical Properties

σ, μ, ε

Survey: DC / IP

- Direct Current (DC) resistivity: sensitive to contrasts in resistivity
- Induced Polarization (IP): sensitive to chargeability
- DC and IP can be acquired in a single survey
- Recovering resistivity from DC data is a non-linear inverse problem
- Recovering chargeability is a linear inverse problem





Century Deposit: geology + physical properties

Mineralized sequence:

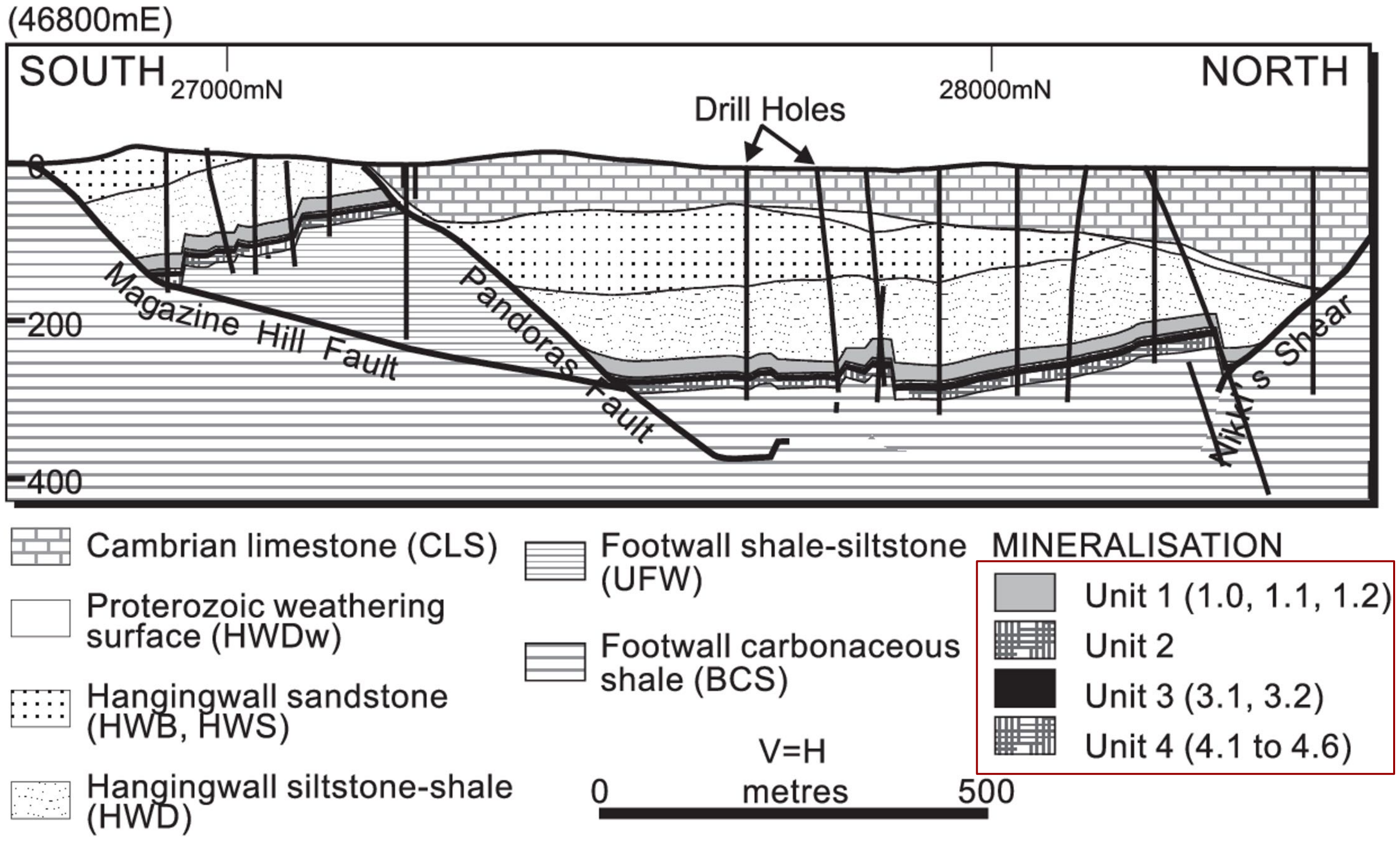
- ~40 m thick
- Pb, Zn within black carbonaceous shales (BCS)

Resistivity

- Provides structural information (faults)
- Needed input to IP

Chargeability

- Associated with mineralization



Yaoguo Li circa 1996

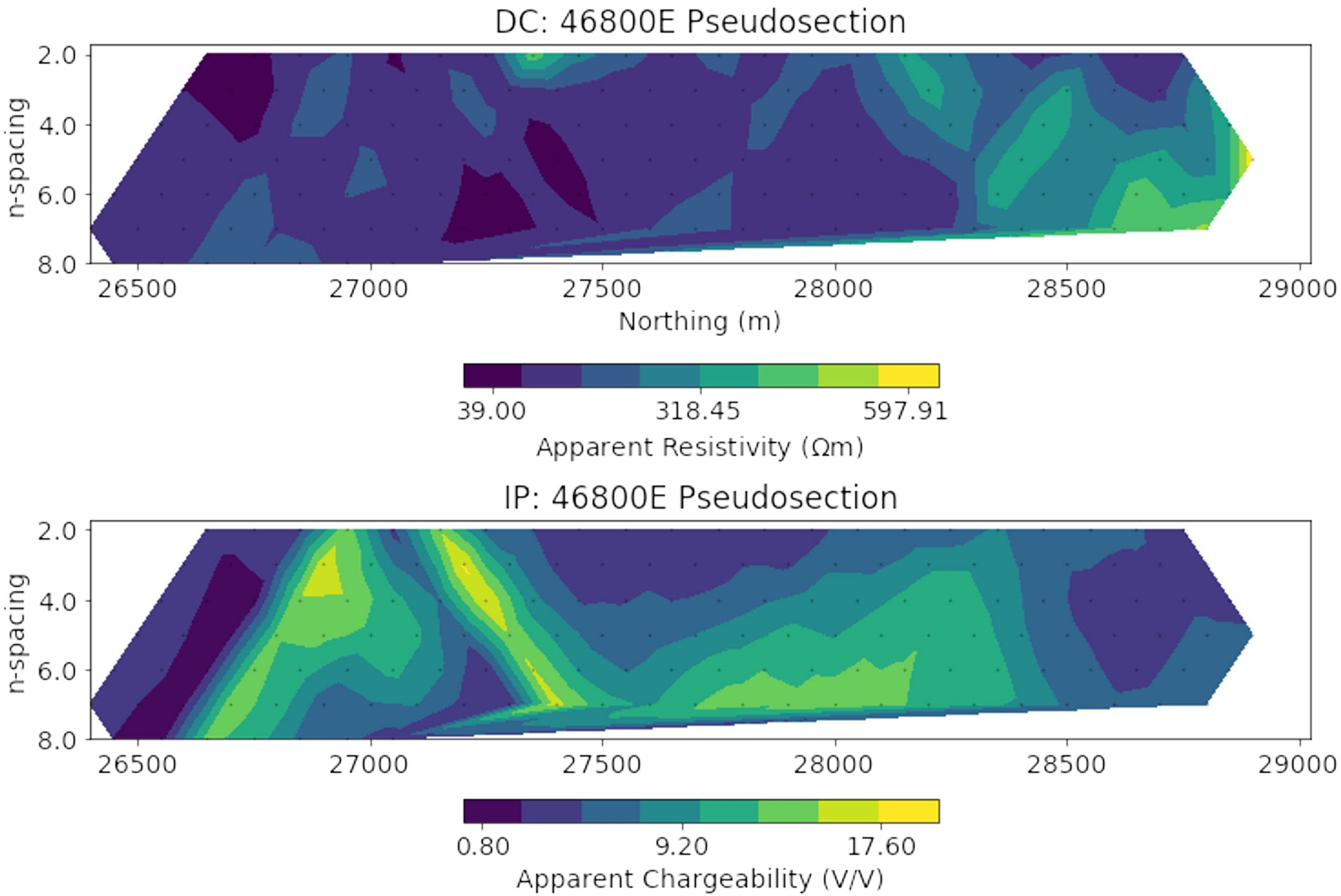
Mutton, 2000

Century Deposit: DC / IP data

DC Resistivity data

IP data

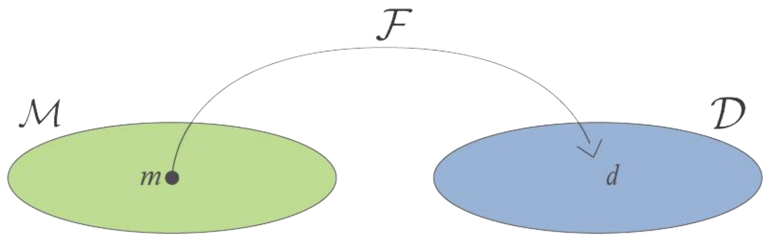
Survey
dipole – dipole
a = 100m
n = 1, 7



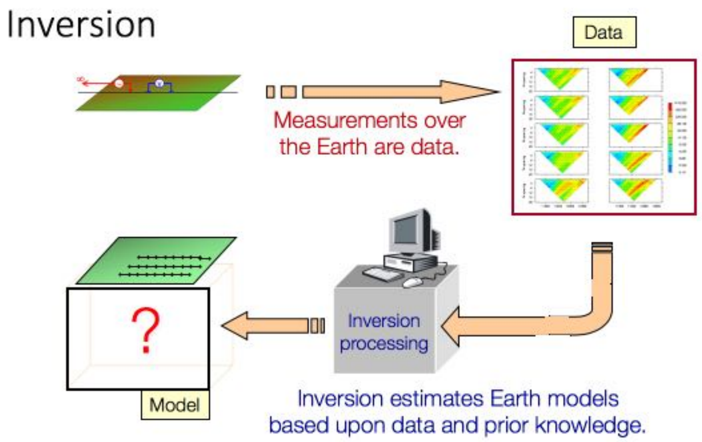
What earth model generated these data?

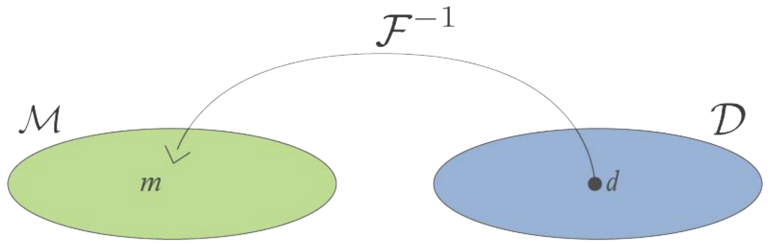
Our statement of the inverse problem

- Given observations: $d_j^{obs}, \quad j = 1, \ldots, N$
  - Uncertainties: ϵ_j
  - Ability for forward modelling: $\mathcal{F}[m] = d$

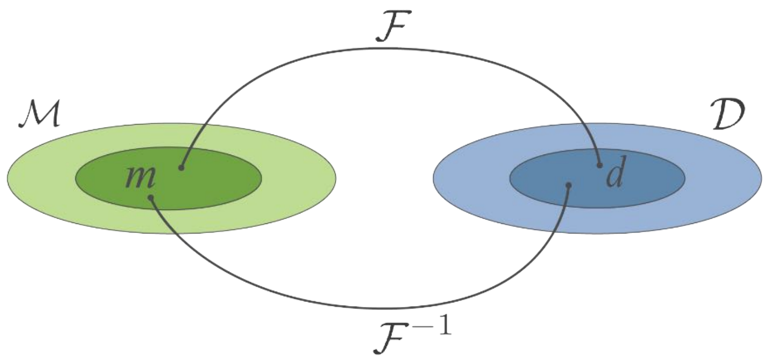


- Find the earth model that gave rise to the data.





Inverse problem



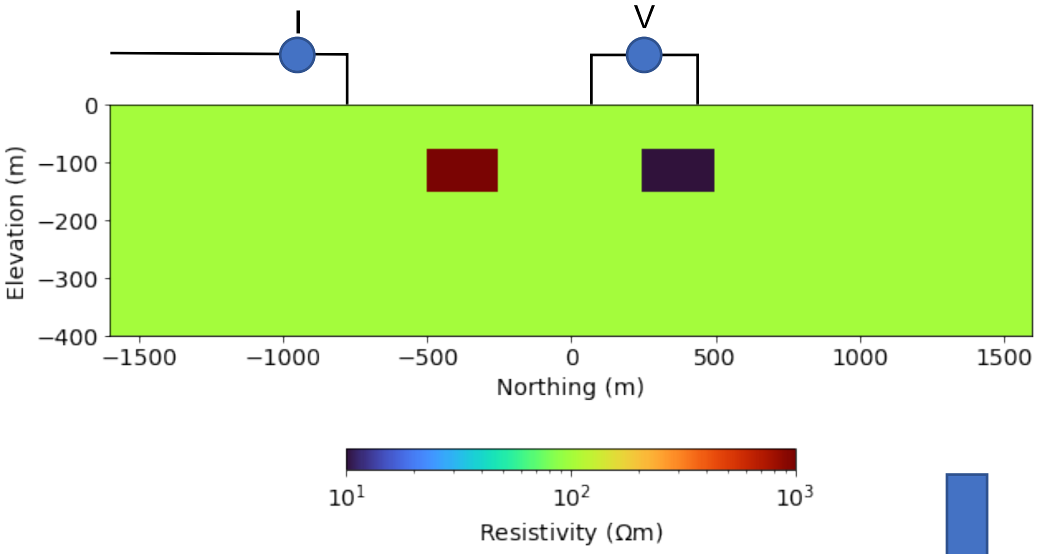
- Non-unique
- Ill-conditioned

The Inverse Problem is ill-posed

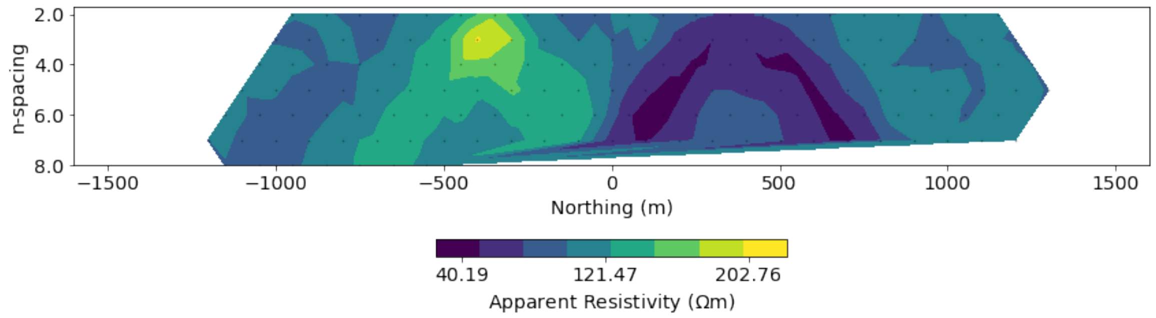
Any inversion approach must address these issues

Example of extreme non-uniqueness

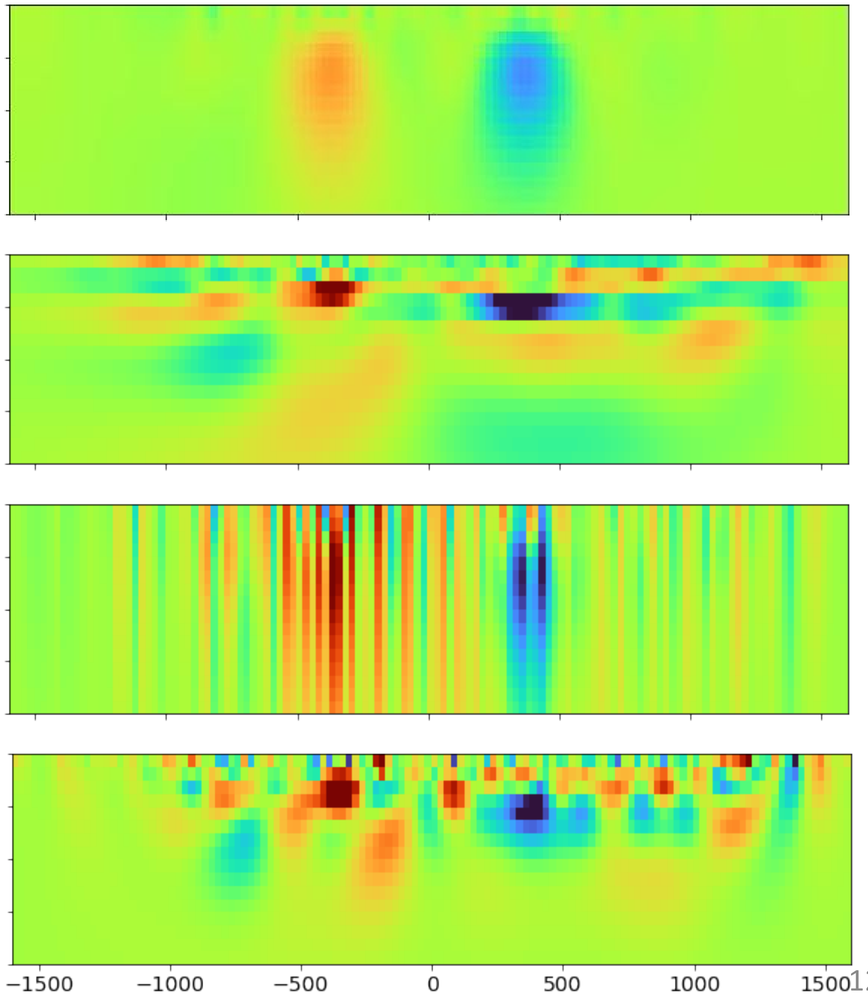
DC experiment



Data



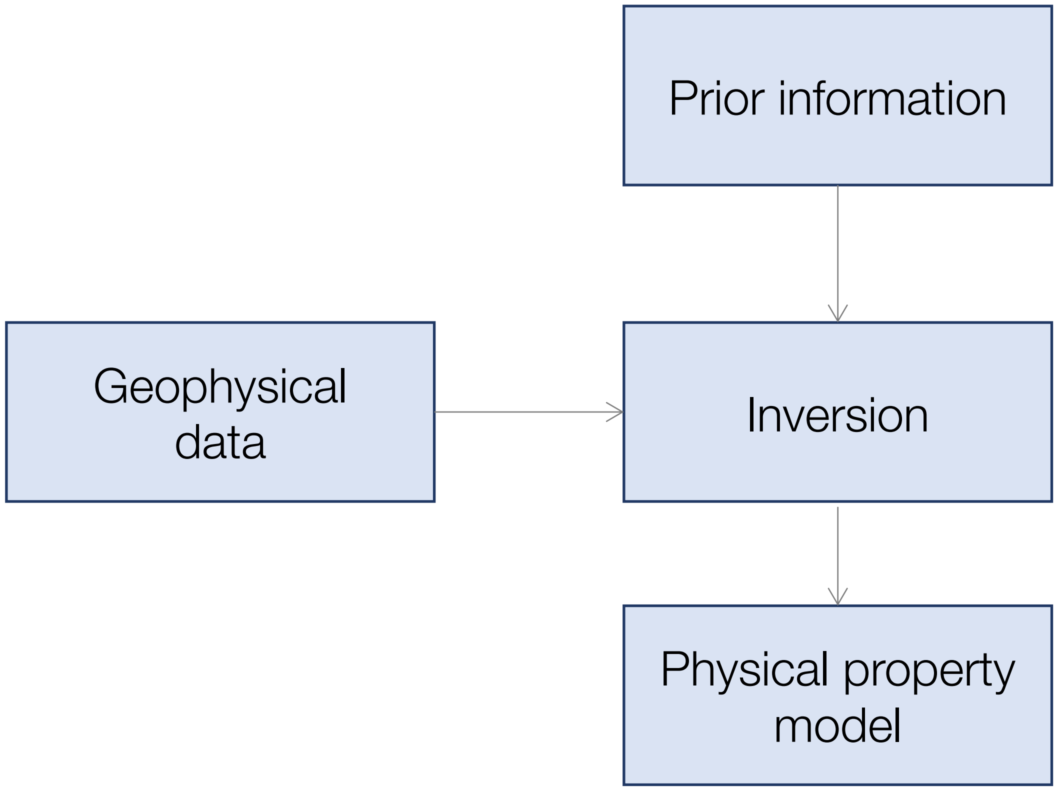
Recovered models



Constraining the inversion

What information is available?

- Geologic structure
- Geologic constraints
- Reference model
- Bounds
- Multiple data sets
- Physical property measurements



How do we achieve our goal?

Need a Framework for Inverse Problem

Tikhonov (deterministic)	Bayesian (probabilistic)

Tikhonov (deterministic)

Find a single "best" solution by solving optimization

$$\text{minimize} \quad \phi = \phi_d + \beta \phi_m$$

$$\text{subject to} \quad m_L < m < m_H$$

$$\begin{cases} \phi_d: \text{ data misfit} \\ \phi_m: \text{ regularization} \\ \beta: \text{ trade-off parameter} \\ m_L, m_H: \text{ lower and upper bounds} \end{cases}$$

Bayesian (probabilistic)

Use Bayes' theorem

$$P(m|d^{obs}) \propto P(d^{obs}|m)P(m)$$

$$\begin{cases} P(m): \text{ prior information about } m \\ P(d^{obs}|m): \text{ probability about the data errors (likelihood)} \\ P(m|d^{obs}): \text{ posterior probability for the model} \end{cases}$$

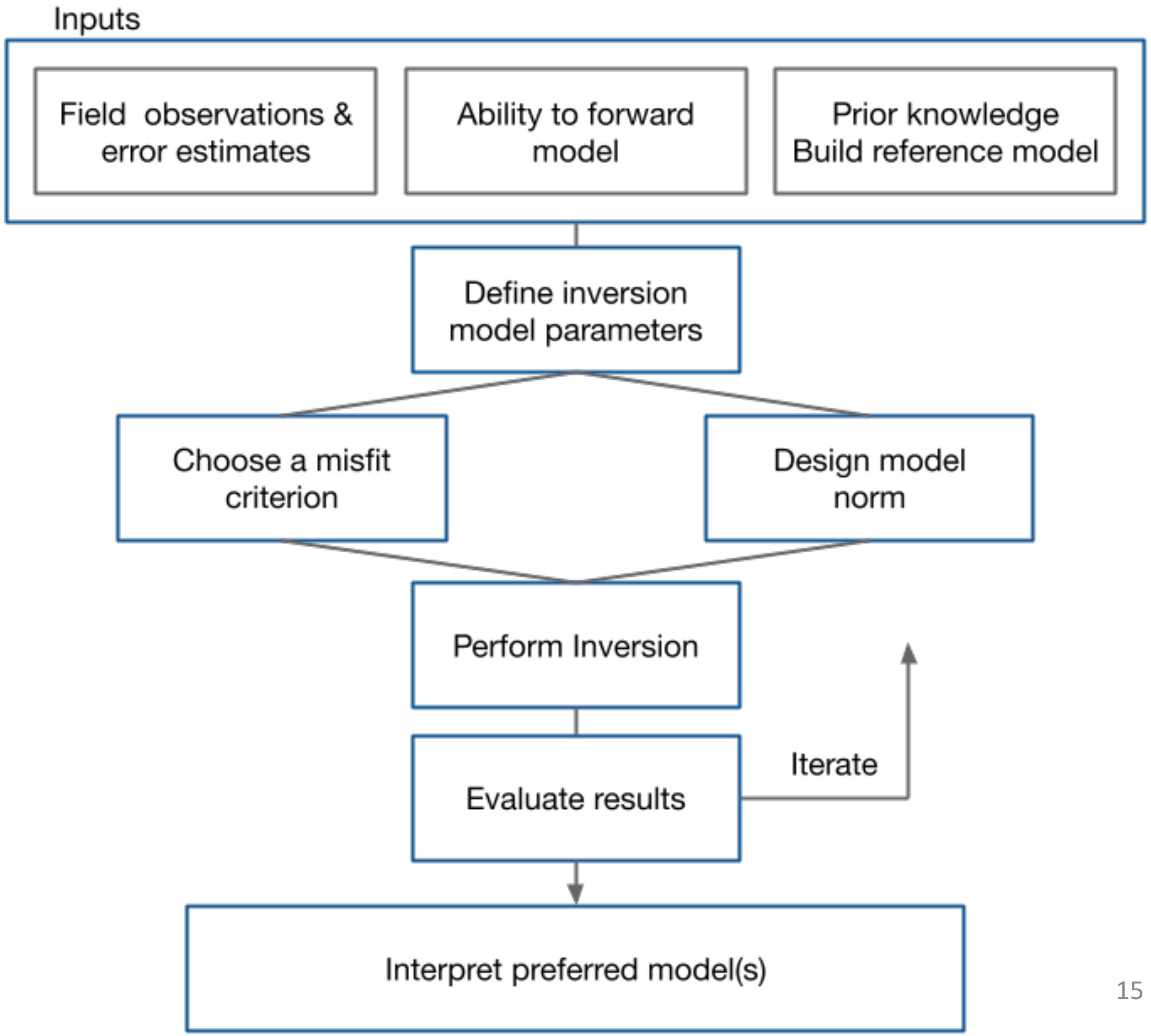
Two approaches:

(a) Characterize $P(m|d^{obs})$

(b) Find a particular solution that maximizes $P(m|d^{obs})$ (MAP: (maximum a posteriori) estimate

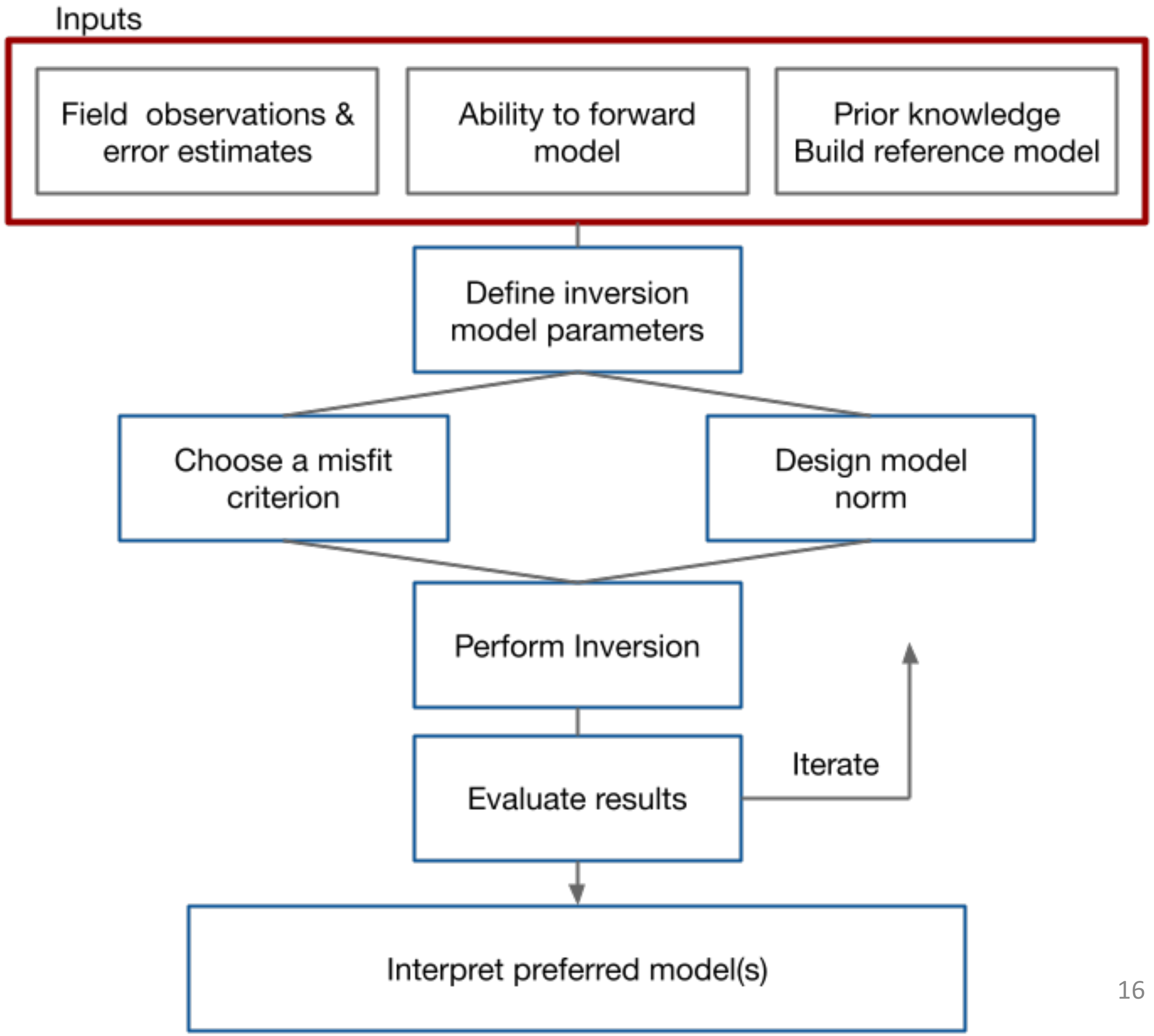
Flow chart for the inverse problem

- Many components to achieving a quality result
- Success is in the details
- Evaluate each step in the box critically before going on



Starting up

- Survey and observations
- What processing has been done?
- Normalization of data
- Ability for forward model
- Assemble geologic, petrophysical information
- Build a reference model
- What is the question you want answered from the inversion?



Forward modelling approaches

Maxwell's equations can be solved as:

- Integral equation (IE)

$$\vec{E}(\vec{r}) = \vec{E}_p(\vec{r}) + \int_V G(\vec{r}, \vec{r}_s)\sigma_a(\vec{r}_s)\vec{E}(\vec{r}_s)dv_s$$

- Differential equation (DE)

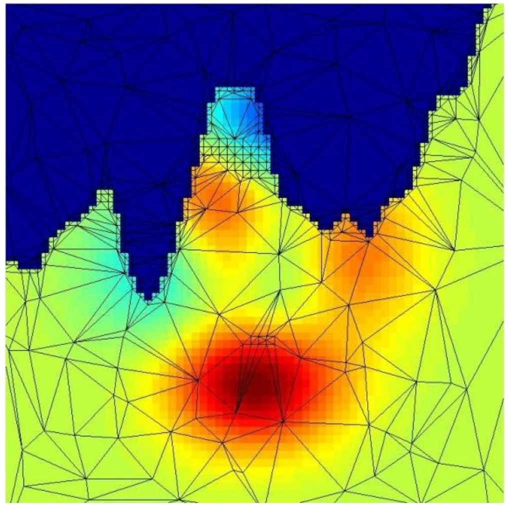
$$\nabla \times \mu^{-1}\nabla \times \vec{E} + \imath\omega\sigma\vec{E} = -\imath\omega\vec{J}_s$$

Desired qualities for a mesh

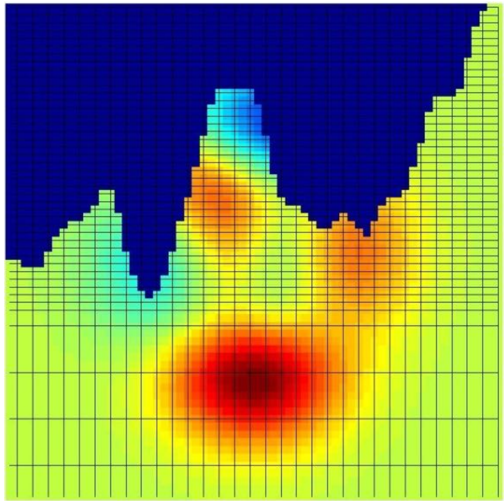
- Conform to structure being modelled
- Small number of cells to reduce computation time
- Be able to discretize equations on the mesh
- “Easy” to solve (sparse matrices)
- Visualize fields and models

What type of mesh?

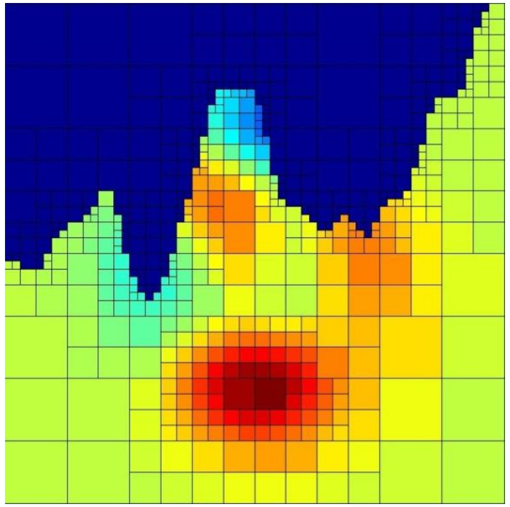
Unstructured



Structured



Semi-structured



To consider:

- Complex geometry
- Matrix size / sparsity
- Visualization
- Complexity of generating
- Ease of programming
- Discretizing to “infinity”
- Cell size / element size changes

Solving differential equations

Problems on unstructured or structured meshes can be solved using

Complexity ↓

- Finite Difference Method (FDM)
- Finite Volume Method (FVM)
- Finite Element Method (FEM)

Solving the forward problem

$$A(m)u = q$$

Methodology depends on A

- **Small:** use SVD or back-slash (and equivalent)
- **Intermediate:** direct solver $A = LL^\top$ or $A = LU$
- **Very big:** iterative techniques

E.g. Airborne TDEM

- Forward problem: 1000 Tx, 50 timesteps → 50,000 solves
- Inversion: 20 GN iterations and using CG solver → 20,000,000

Forward problem must be efficient; need lots of processors for big problems

Sanity checks for forward modelling

- Test numerical results against a (semi-) analytic solution (eg. halfspace, sphere)
- Estimate numerical modelling errors (this can useful later when assigning "uncertainties" in the inversion)
- Forward model your reference model and see how them match the data. Check: Normalization errors? Coordinate system? …

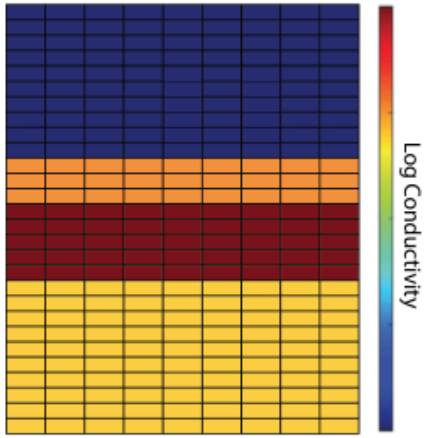
Inversion model parameters

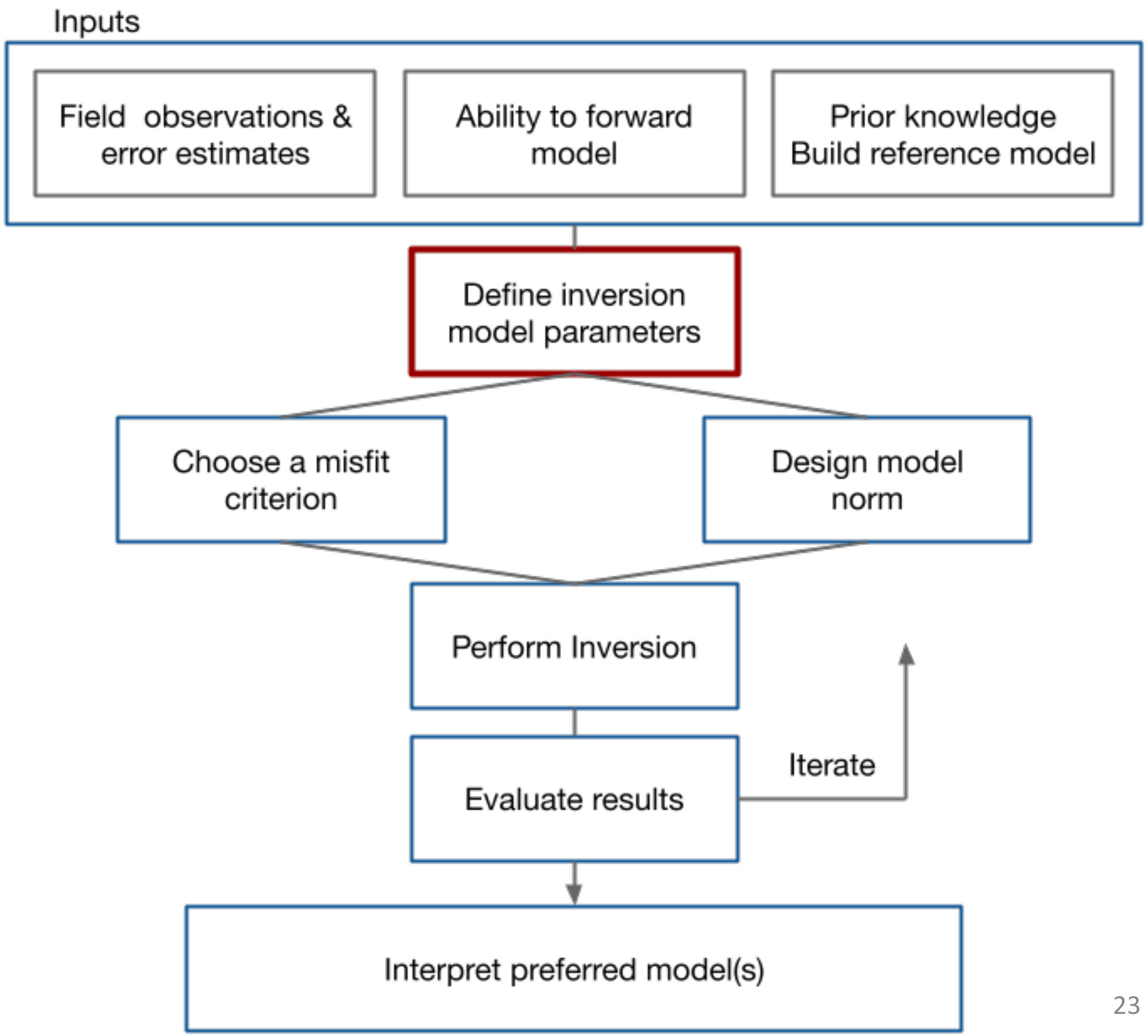
- In the forward problem

$$d = \mathcal{F}[m]$$

m is our sought function (conductivity, density, ….)

- Inverse problem: we have options (eg log sigma, parametric ….)





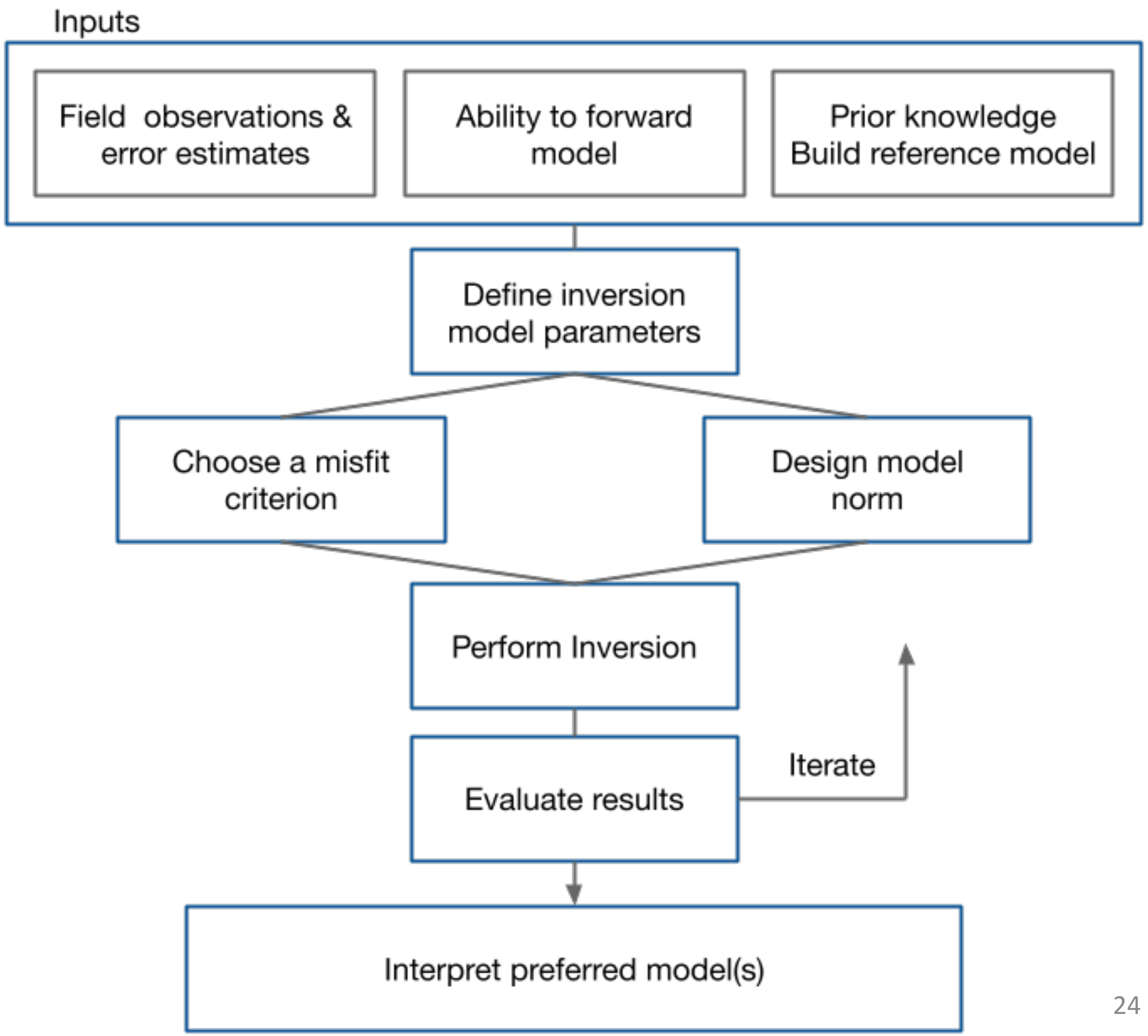
Inversion as an optimization problem

- Find a single "best" solution by solving optimization

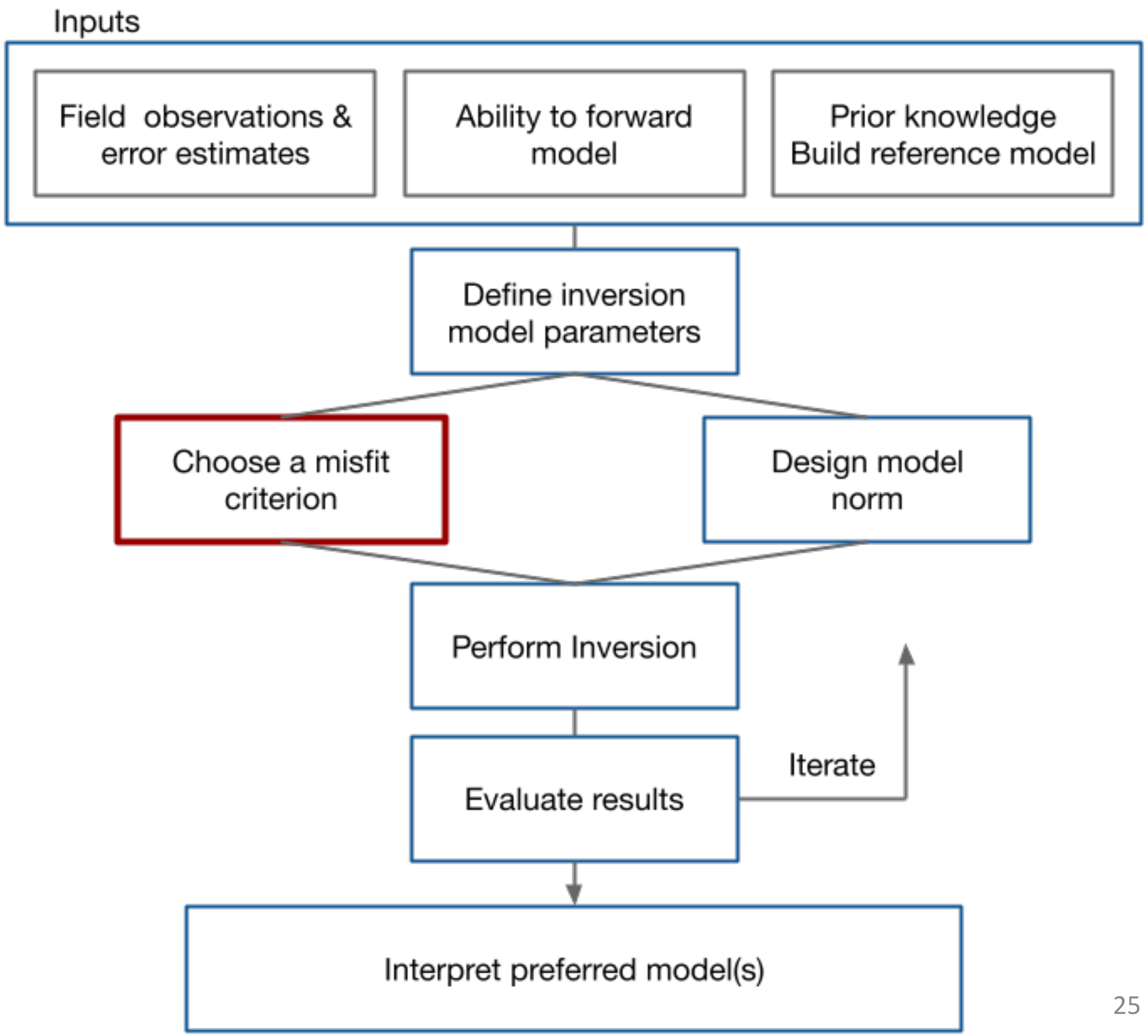
$$\text{minimize} \quad \phi = \phi_d + \beta\phi_m$$

$$\text{subject to} \quad m_L \leq m \leq m_U$$

$$\phi_d : \text{ data misfit}$$
$$\phi_m : \text{ model norm}$$
$$\beta : \text{ trade-off parameter}$$
$$m_L, m_U : \text{ lower and upper bounds}$$



Flow chart for the Inverse problem



Dealing with uncertainties

Observed datum

$$d_j^{obs} = F_j(m) + n_j$$

Noise n_j includes

- Modelling errors
  - dimensionality errors (1D v. 3D)
  - incomplete physics
  - discretization errors
- Noise on data
  - instrument / sensor noise
  - survey parameter errors
  - wind …

True statistics of "noise" is complicated.
In practice, assume errors are Gaussian
$\mathcal{N}(0, \epsilon_j)$

Dealing with uncertainties

Consider $\text{random variable, } x_j \in \mathcal{N}(0,1)$

Define $$\chi_N^2 = \sum_{j=1}^{N} x_j^2$$

Chi-squared statistic with N degrees of freedom

$$\begin{cases} \text{Expected value: } E(\chi_N^2) = N \\ \text{Variance: } \text{Var}(\chi_N^2) = 2N \\ \text{Standard deviation: } \text{std}(\chi_N^2) = \sqrt{2N} \end{cases}$$

Misfit function

Crucial steps for any misfit:

(1) Specify the metric used

(2) Determine target misfit

We use L_2 norm (least squares statistic)

Define data misfit: $\phi_d = \sum_{j=1}^{N} \left(\frac{F_j(m) - d^{obs}}{\epsilon_j} \right)^2$

Define $\mathbf{W}_d = \mathbf{diag}(1/\epsilon_1, \ldots, 1/\epsilon_N)$

$$\phi_d = \|\mathbf{W}_d(F[\mathbf{m}] - \mathbf{d}^{obs})\|_2^2$$

$$E[\phi_d] \simeq N$$

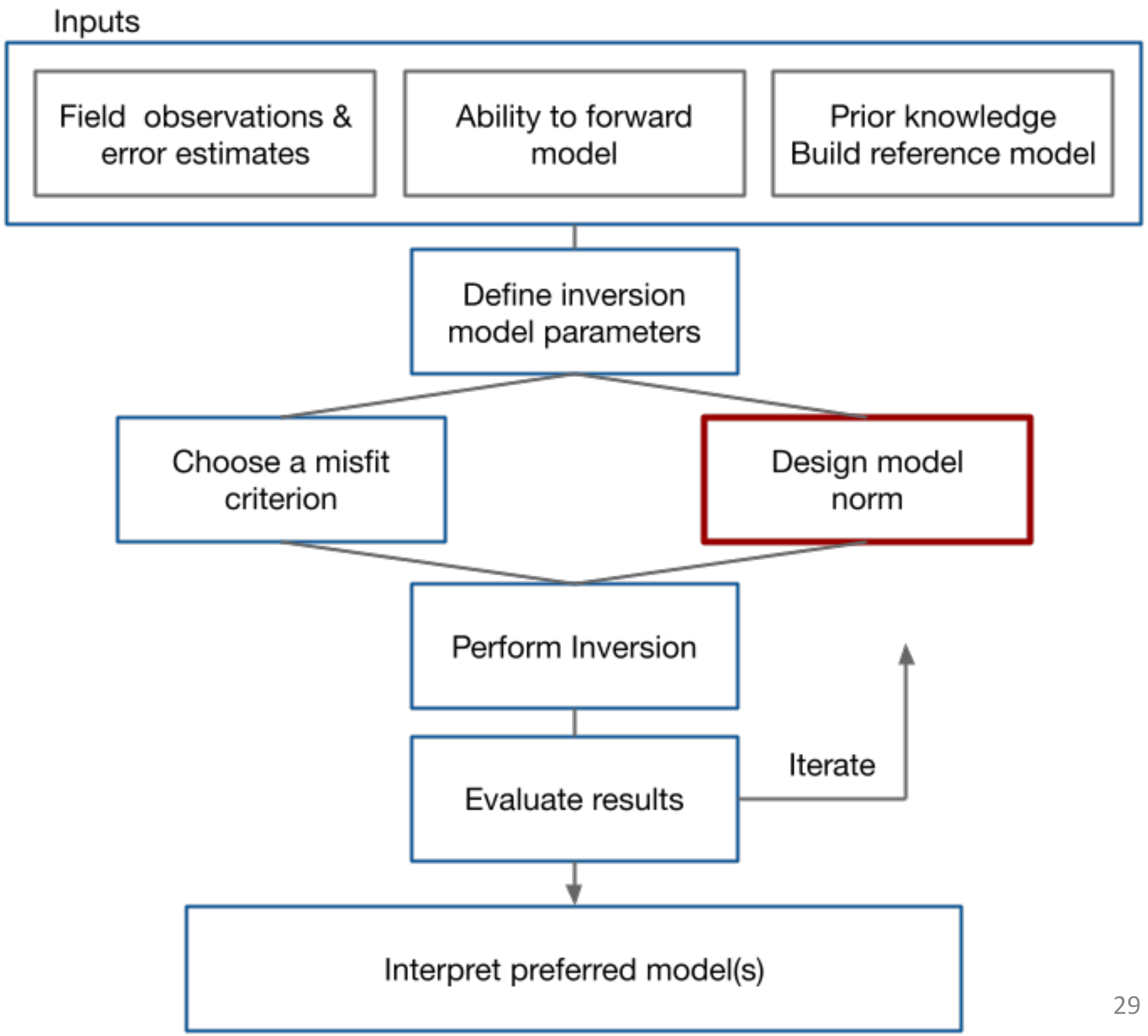
ϕ_d is now a χ^2_N variable

Reality: we do not know uncertainties

Try:

$$\epsilon_j = \%|d_j^{obs}| + \text{floor}$$

Flow chart for the Inverse problem



Model norms

First define our model norms as functions and then discretize

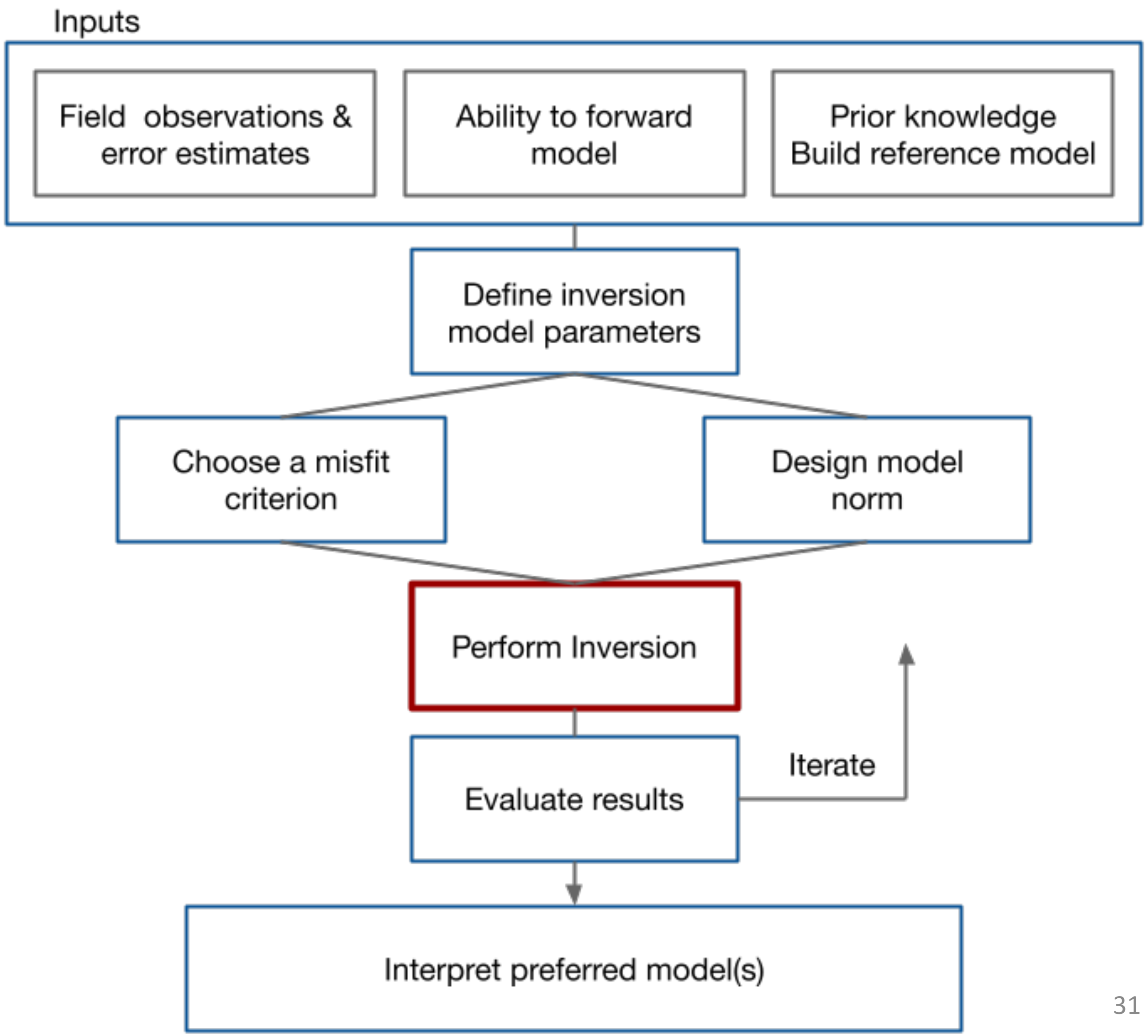
Smallest model: $$\phi_m = \int (m - m_{ref})^2 dx$$

Flattest model: $$\phi_m = \int \left(\frac{dm}{dx}\right)^2 dx$$

Combination: $$\phi_m = \alpha_s \int (m - m_{ref})^2 dx + \alpha_x \int \left(\frac{dm}{dx}\right)^2 dx$$

Discretize: $$\phi_m = \alpha_s \|\mathbf{W}_s(\mathbf{m} - \mathbf{m}_{ref})\|_2^2 + \alpha_x \|\mathbf{W}_x(\mathbf{m})\|_2^2$$

Flow chart for the Inverse problem

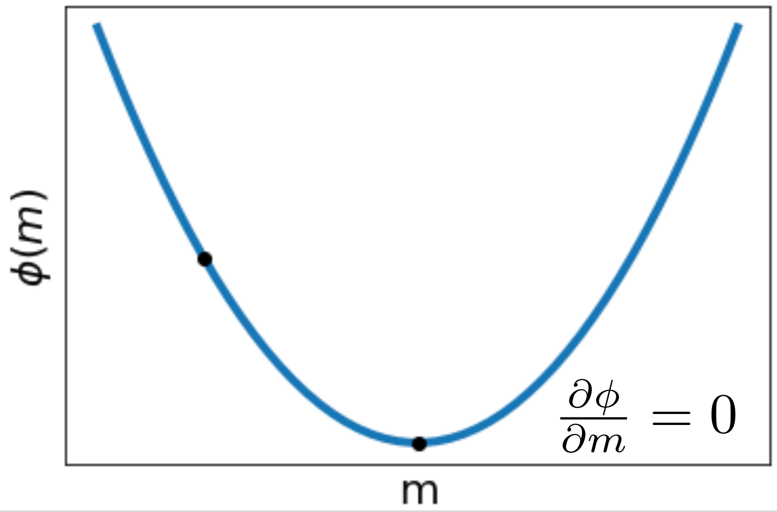


Perform inversion: Linear Forward problem

Linear problem $\mathcal{F}[m] = d$ → $\mathbf{G}\mathbf{m} = \mathbf{d}$

$$\phi(\mathbf{m}) = \frac{1}{2}\|\mathbf{W}_d(\mathbf{G}\mathbf{m} - \mathbf{d}^{obs})\|^2 + \frac{\beta}{2}\|\mathbf{W}_m(\mathbf{m} - \mathbf{m}_{ref})\|^2$$

Quadratic objective function (for a single variable)



$\mathbf{g} = \nabla_m \phi$ $\quad$ $\mathbf{g} = \mathbf{G}^\top\mathbf{W}_d^\top\mathbf{W}_d(\mathbf{G}\mathbf{m} - \mathbf{d}^{obs}) + \beta\mathbf{W}_m^\top\mathbf{W}_m(\mathbf{m} - \mathbf{m}_{ref})$

$\mathbf{g} = 0$ $\quad$ $(\mathbf{G}^\top\mathbf{W}_d^\top\mathbf{W}_d\mathbf{G} + \beta\mathbf{W}_m^\top\mathbf{W}_m)\mathbf{m} = \mathbf{G}^\top\mathbf{W}_d^\top\mathbf{W}_d\mathbf{d}^{obs} + \beta\mathbf{W}_m^\top\mathbf{W}_m\mathbf{m}_{ref}$

$$\mathbf{H}\mathbf{m} = \mathbf{b}$$

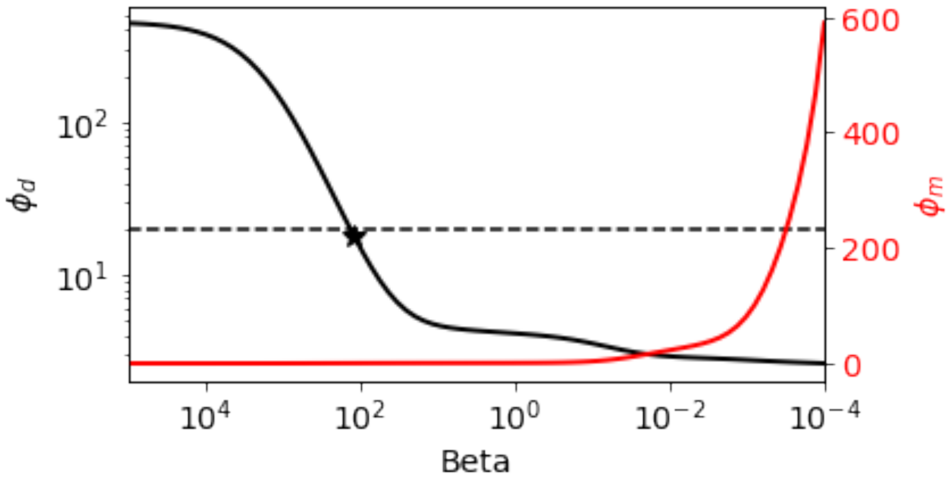
$$\begin{cases}\mathbf{H} \in \mathbb{R}^{M\times M} \text{ is full rank} \\ \mathbf{m}, \mathbf{b} \in \mathbb{R}^M\end{cases}$$

$$\mathbf{m} = \mathbf{H}^{-1}\mathbf{b}$$

Role of beta

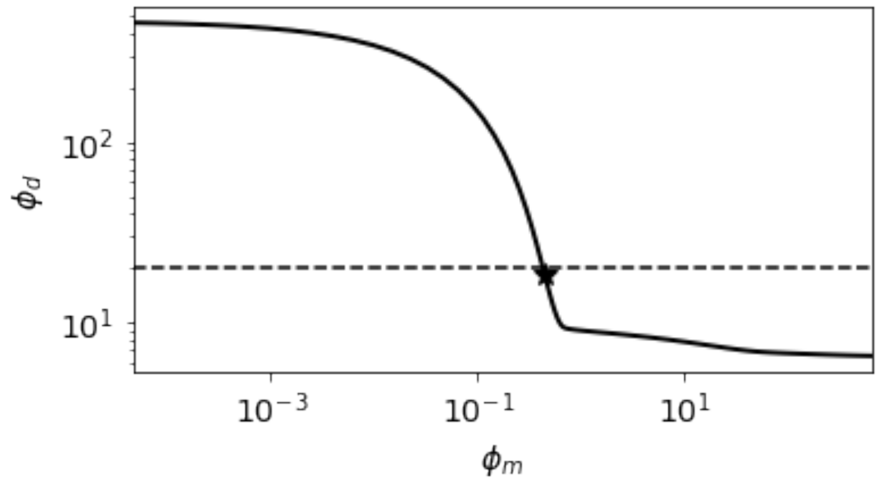
$$\phi(m) = \phi_d(m) + \beta\phi_m(m)$$

$$\beta \to 0 \;: \quad \phi \sim \phi_d$$
$$\beta \to \infty : \quad \phi \sim \phi_m$$



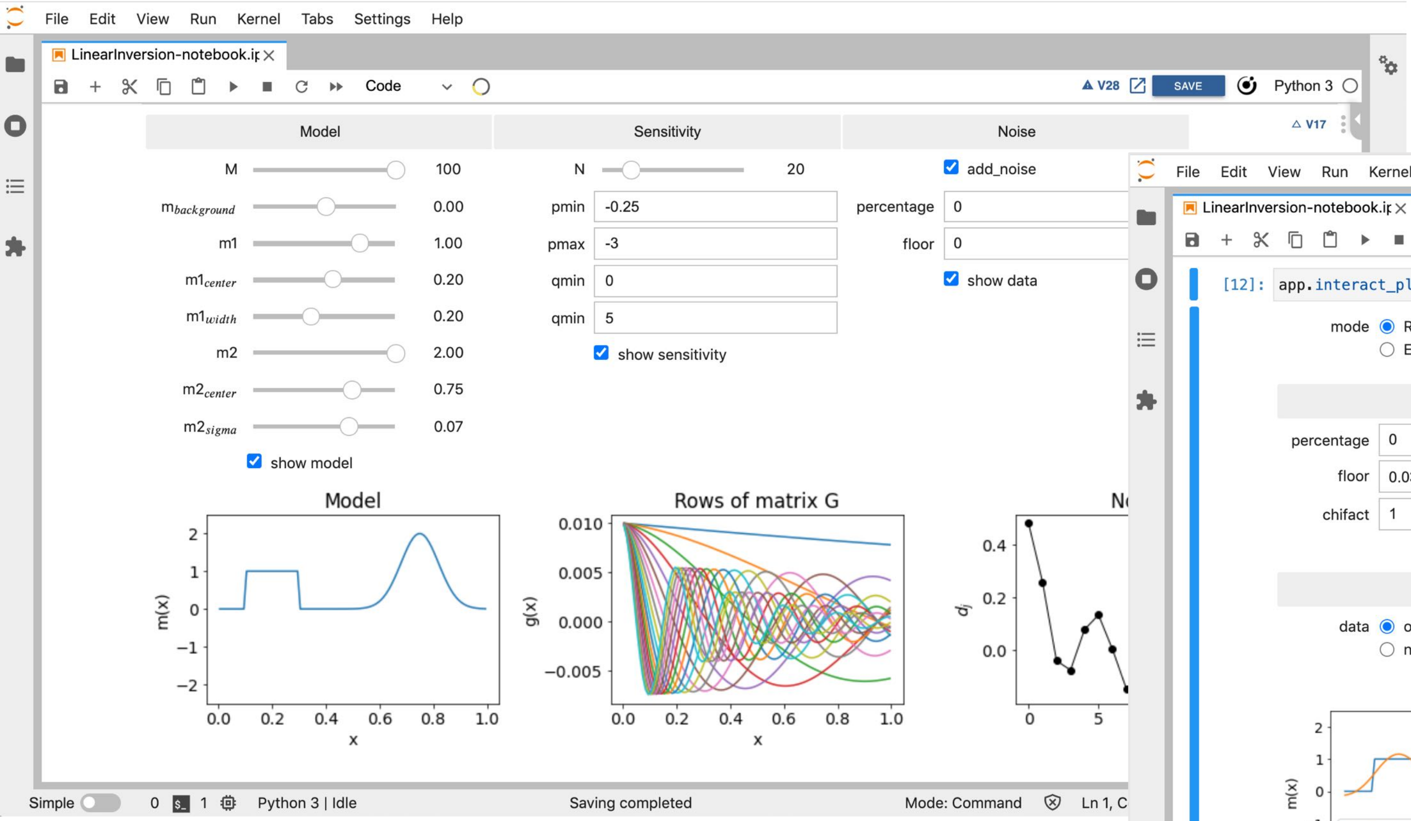
Tikhonov Curve

- Desired misfit $\phi_d^* \simeq N$
- Choose β such that $\phi_d(m) = \phi_d^*$

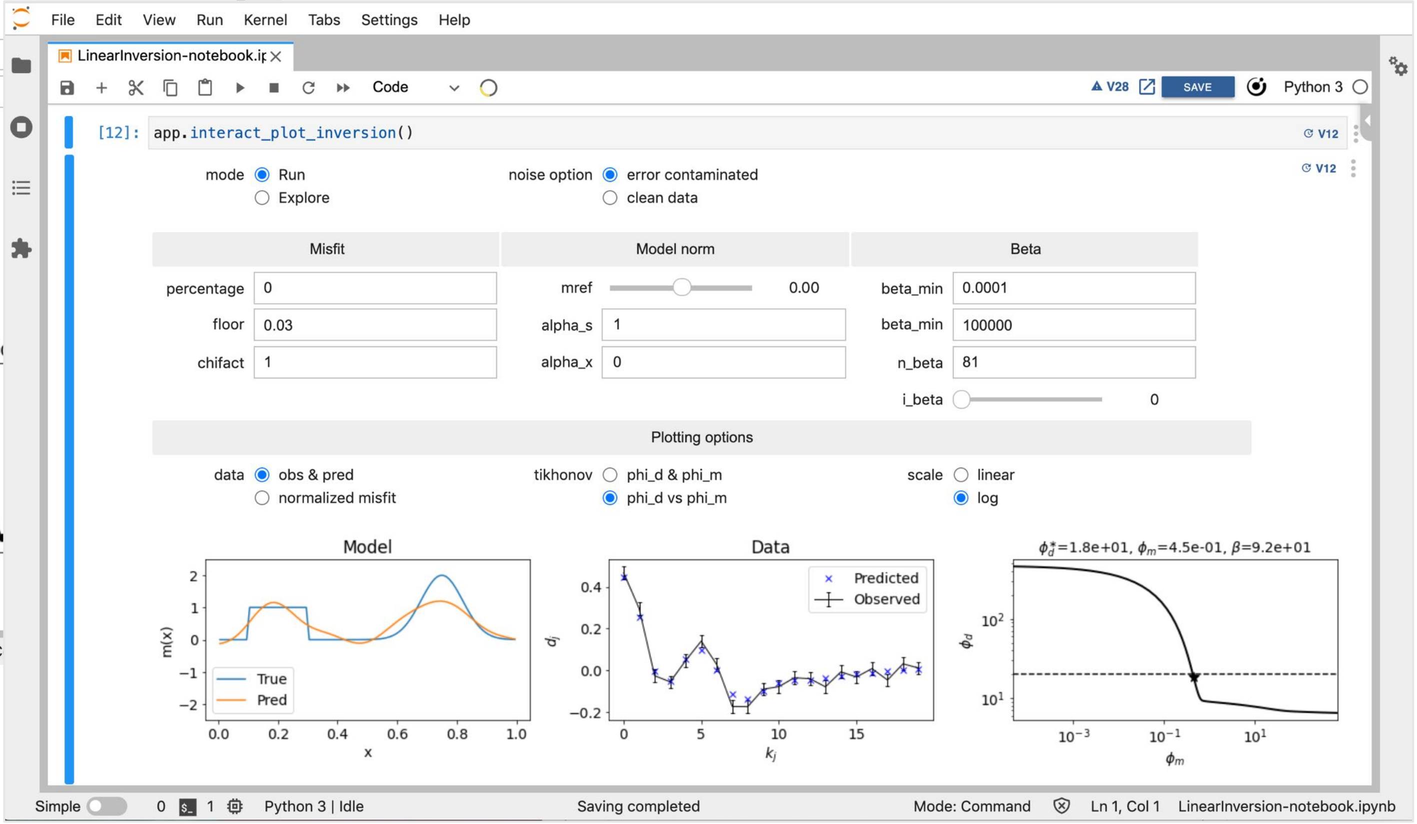


Linear inversion app (demo)

Develop survey



Run inversion



curvenote.com/@geosci/inversion-module/linearinversion

Linear IP problem

Linear model for IP (Seigel, 1959)

- Chargeability: η
- Effect increases resistivity

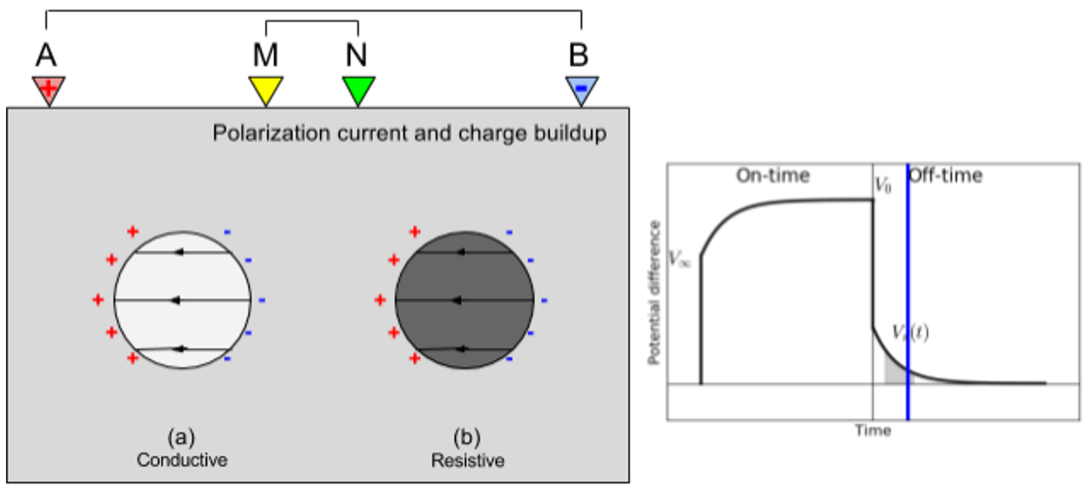
$$\rho_\eta = \rho \frac{1}{1-\eta} \quad \eta \in [0, 1)$$

An IP datum can be written as:

$$d_i^{IP} = \sum_{j=1}^{M} J_{ij}\eta_j \quad i = 1, ..., N$$

Where $J_{i,j}$ are the sensitivities for the DC problem

$$J_{i,j} = \frac{\partial \log \phi^i}{\partial \log \rho_j}$$

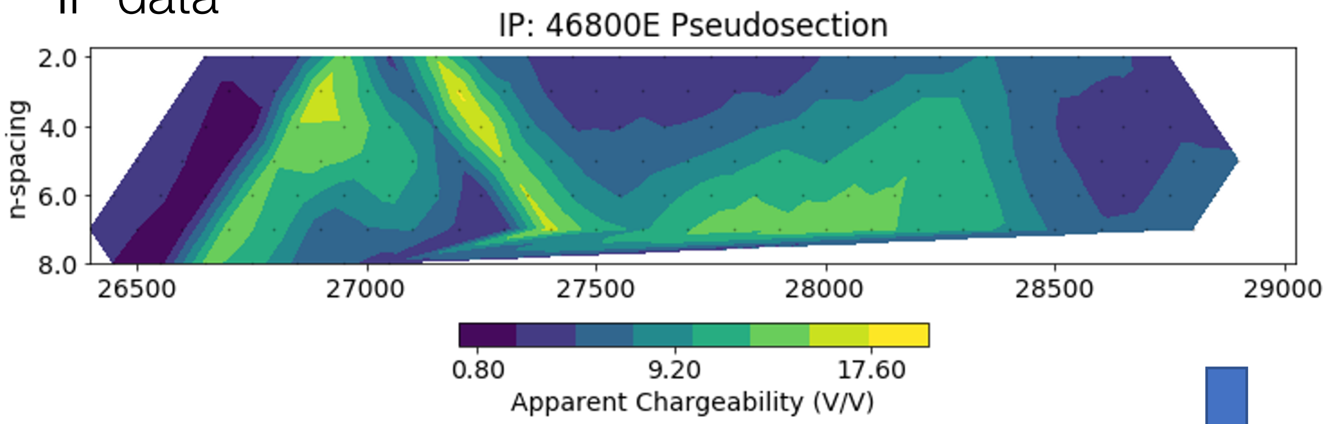


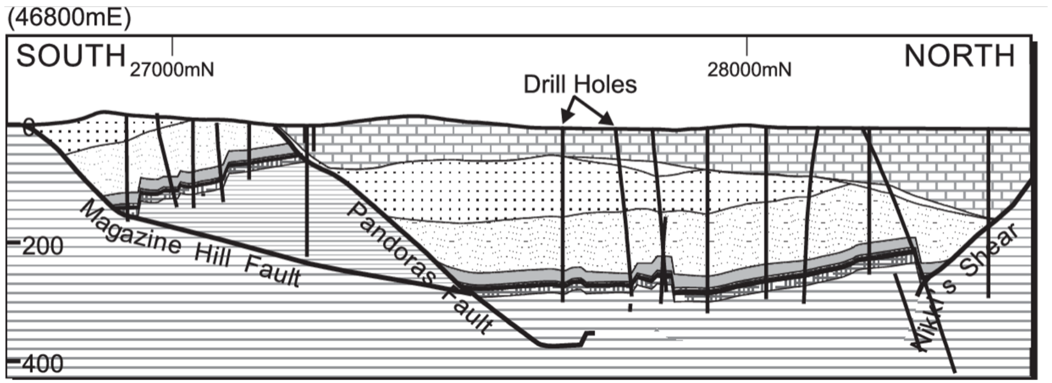
Governing linear equation

$$\mathbf{d^{IP}} = \mathbf{J}\eta$$

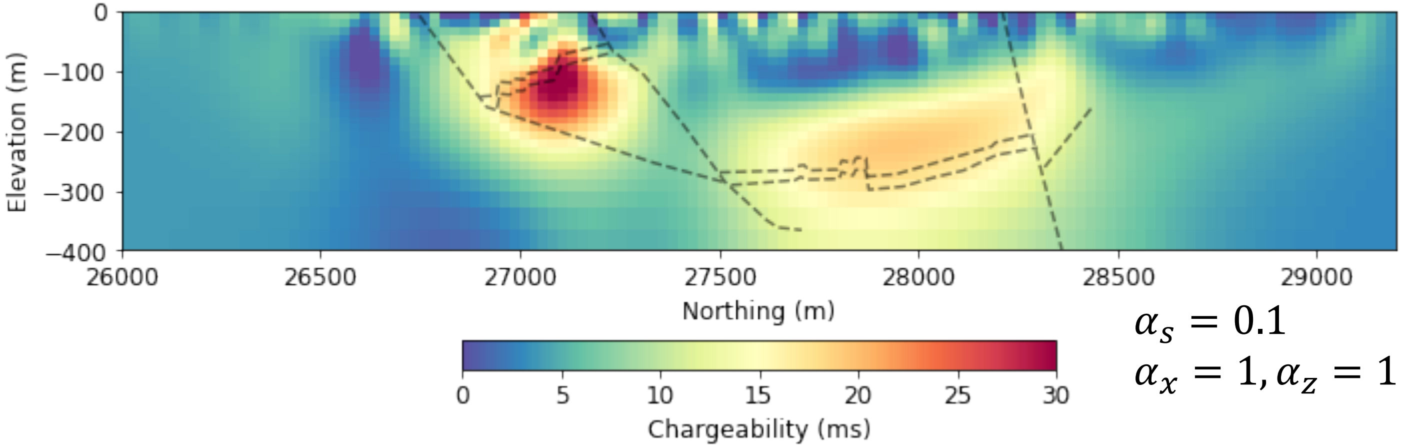
Field example: IP Century deposit

IP data





Recovered chargeability (without positivity)



$\alpha_s = 0.1$

$\alpha_x = 1, \alpha_z = 1$

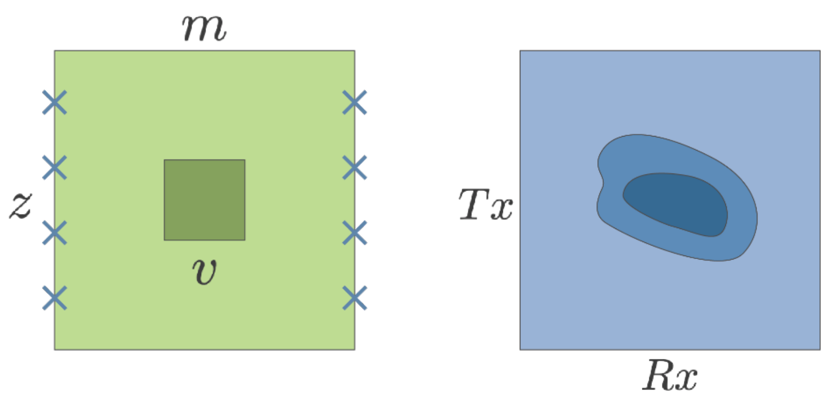
Non-linear inversion

Non-linear inversion

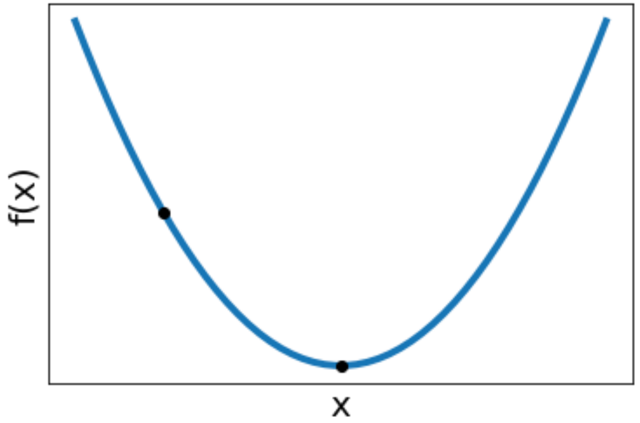
- Inverse problem

$$\text{minimize} \quad \phi = \phi_d + \beta\phi_m$$

$$\phi_d = \sum_{j=1}^{N} \left(\frac{\mathcal{F}_j(m) - d_j^{obs}}{\epsilon_j} \right)^2$$



- For linear problem: $d = Gm$
  - And quadratic regularization: $\int_v (m - m_{\text{ref}})^2 dv$
  - This is quadratic so we can solve in one step

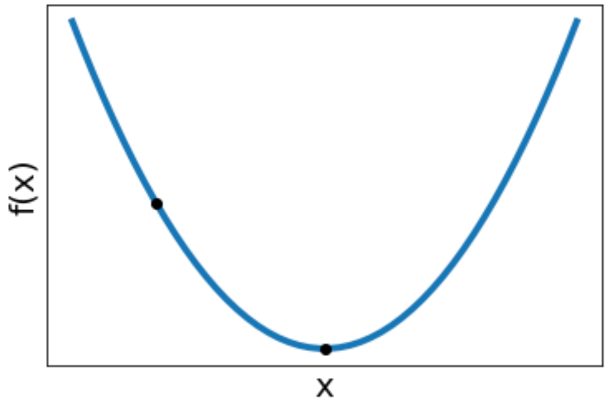


- Problem becomes non-linear if:

(i) $\mathcal{F}[m]$ is non-linear
(ii) ϕ_d is not l_2 (e.g. $\sum \left| \frac{\mathcal{F}_i[m] - d_i}{\epsilon_i} \right|$)
(iii) ϕ_m is not quadratic

Non-linear optimization

- Single variable x: $\text{minimize } f(x)$ $f(\cdot)$: function



- Case I: f is quadratic

$$f(x) = \frac{1}{4}x^2 - 3x + 9 = (\frac{1}{2}x - 3)^2$$

$$f'(x) = (\frac{1}{2}x - 3) = 0 \quad \Rightarrow \quad x = 6$$

Minimum: $f'(x) = 0$
$f''(x) > 0$

- Suppose

$$f(x) = (\frac{1}{2}x - 3)^2 + ax^3 + bx^4$$

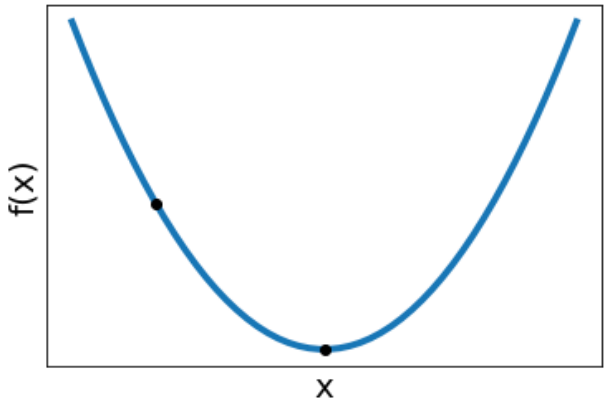
Non-linear optimization

- Single variable x: $\text{minimize } f(x)$ $f(\cdot)\text{: function}$

- Case I: f is quadratic

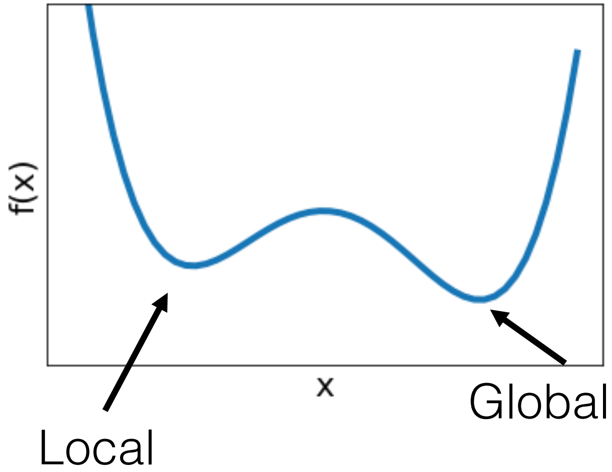
$$f(x) = \frac{1}{4}x^2 - 3x + 9 = (\frac{1}{2}x - 3)^2$$

$$f'(x) = (\frac{1}{2}x - 3) = 0 \Rightarrow x = 6$$



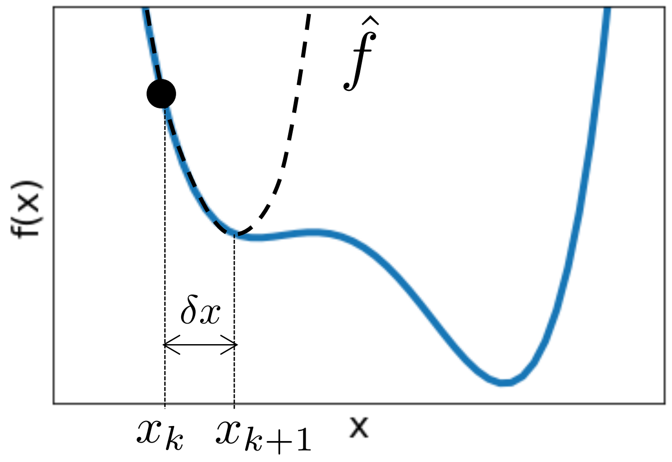
- Suppose

$$f(x) = (\frac{1}{2}x - 3)^2 + ax^3 + bx^4$$



Non-linear optimization

- Newton's Method
  i. Begin with x_k
  ii. Solve a local quadratic for δx
  iii. $x_{k+1} = x_k + \delta x$



Local Quadratic: $\hat{f}(x_k + \delta x) = f(x_k) + f'(x_k)\delta x + \frac{1}{2} f''(x_k)\delta x^2 + \mathcal{O}(\delta x^3)$

δx that minimizes $\hat{f}(x_k + \delta x)$ ⇒

$$f''(x_k)\delta x = -f'(x_k)$$

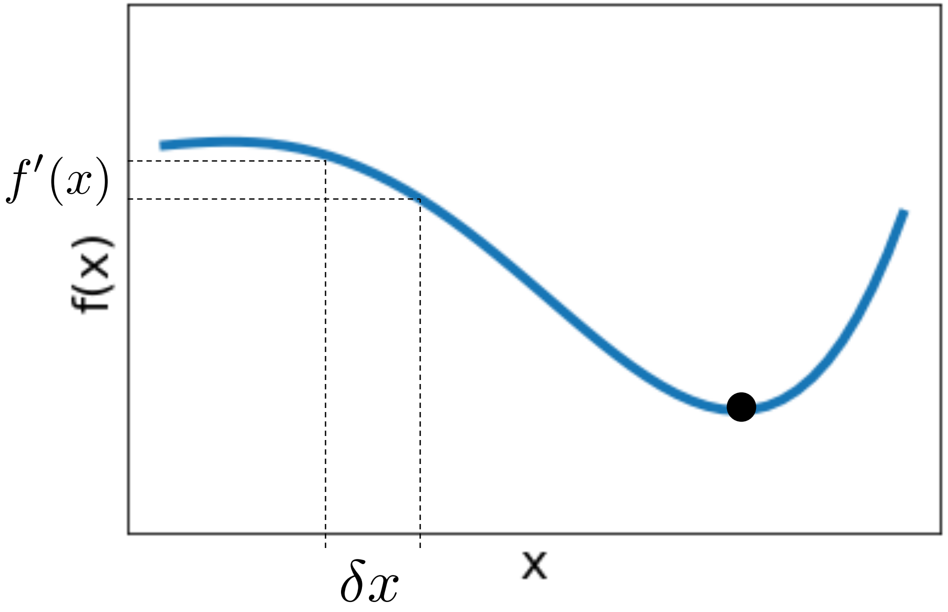
$$\text{or } \delta x = -\frac{f'(x_k)}{f''(x_k)}$$

Convergence conditions

$$f'(x) < \text{tolerance}$$

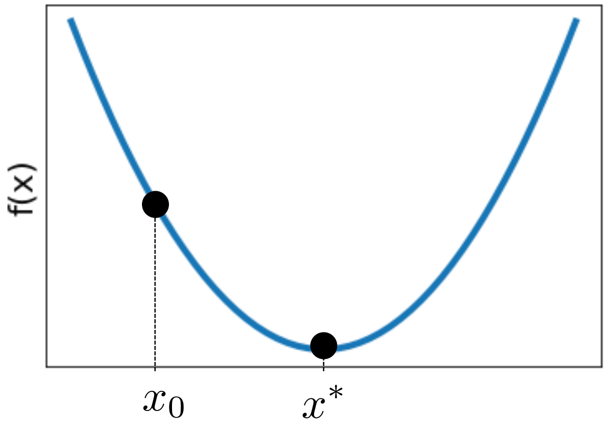
$$\|\delta x\| < \text{tolerance}$$

$$f''(x) > 0$$



Summary: Newton's method

Linear

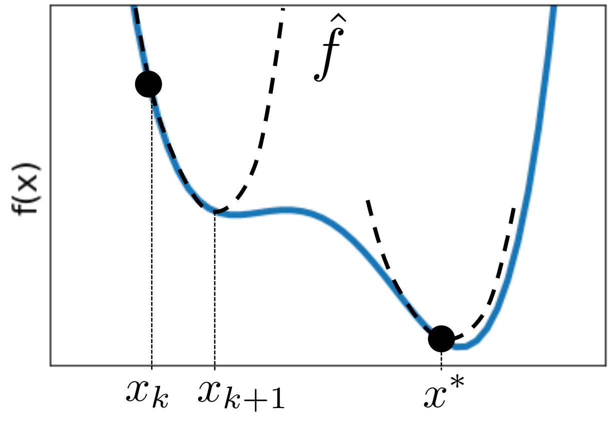


$$f''(x)\delta x = -f'(x)$$

$$x^* = -\frac{f'(x)}{f''(x)}$$

Solution in one step

Non-linear



$$f''(x_k)\delta x = -f'(x_k)$$

$$\delta x = -\frac{f'(x_k)}{f''(x_k)}$$

$$x_{k+1} = x_k + \alpha \delta x \qquad \alpha < 1$$

Iterate to convergence

Multivariate functions

Minimize $\phi(m)$ $m \in \{m_1, m_2, ..., m_M\}$

Taylor expansion

$$\phi(m + \delta m) = \phi(m) + (\nabla_m \phi(m))^T \delta m + \frac{1}{2} \nabla_m \nabla_m \phi(m) \delta m + \mathcal{O}(\delta m^3)$$

Note similarity to single variable

$$f(x + \delta x) = f(x) + f'(x) \delta x + \frac{1}{2} f''(x) \delta x^2 + \mathcal{O}(\delta x^3)$$

Define

Gradient: $g(m) = \nabla_m \phi = \begin{pmatrix} \frac{\partial \phi}{\partial m_1} \\ \vdots \\ \frac{\partial \phi}{\partial m_M} \end{pmatrix}$ $\qquad g \in \mathbb{R}^M$

Hessian:

$$H(m) = \nabla_m \nabla_m \phi = \begin{pmatrix} \frac{\partial^2 \phi}{\partial m_1^2} & \frac{\partial^2 \phi}{\partial m_1 \partial m_2} & \cdots & \frac{\partial^2 \phi}{\partial m_1 \partial m_M} \\ \frac{\partial^2 \phi}{\partial m_2 \partial m_1} & \frac{\partial^2 \phi}{\partial m_2^2} & \cdots & \frac{\partial^2 \phi}{\partial m_2 \partial m_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \phi}{\partial m_M \partial m_1} & \frac{\partial^2 \phi}{\partial m_M \partial m_2} & \cdots & \frac{\partial^2 \phi}{\partial m_M^2} \end{pmatrix} \qquad H \in \mathbb{R}^{M \times M}$$

Symmetric

$$H_{i,j} = \frac{\partial^2 \phi}{\partial m_i \partial m_j}$$

Minimum defined:

$g(m^*) = 0$

$H(m^*)$ is positive definite

Finding a solution

(i) Begin with $m^{(k)}$

(ii) Solve $H\big(m^{(k)}\big)\delta m = -g\big(m^{(k)}\big)$ c.f. $\{f''(x)\delta x = -f'(x)\}$

(iii) $m^{(k+1)} = m^{(k)} + \alpha\delta m$

Our inversion

Minimize: $$\phi(m) = \frac{1}{2}\|\mathcal{F}[m] - d^{obs}\|^2 + \frac{\beta}{2}\|m\|^2$$

Gradient: $$g(m) = \nabla_m \phi = J^T\left(\mathcal{F}[m] - d^{obs}\right) + \beta m$$

Hessian: $$H(m) = \nabla_m g(m) = J^T J + (\nabla_m J)^T \left(\mathcal{F}[m] - d^{obs}\right) + \beta$$

neglect

Sensitivity:

$$\nabla_m \mathcal{F}(m) = J$$

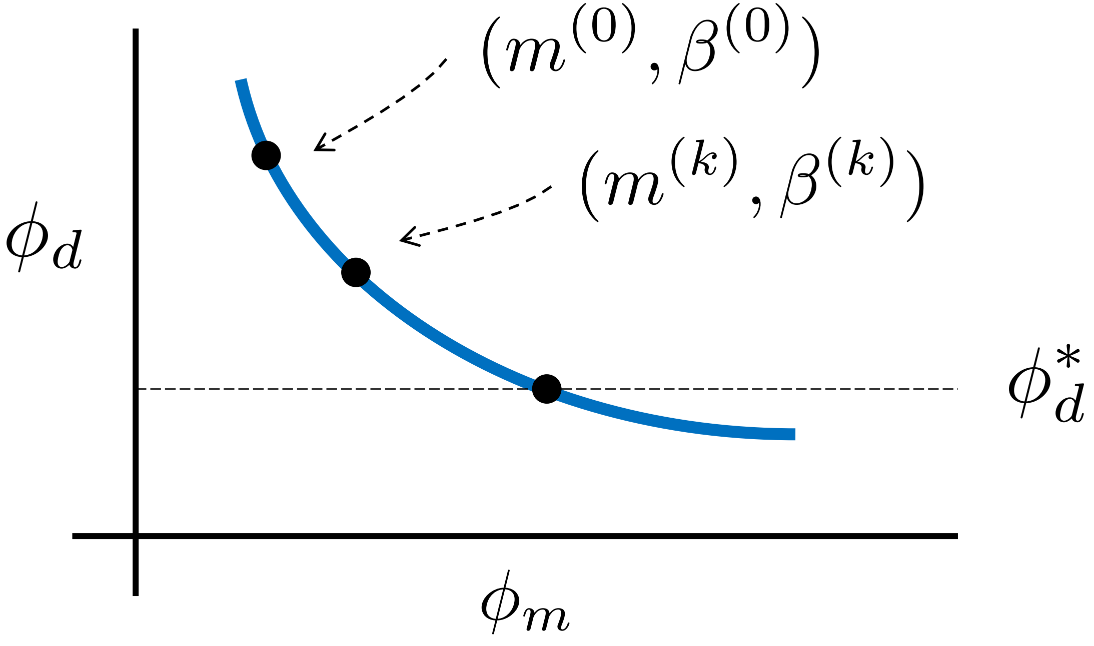
$$J_{ij} = \frac{\partial \mathcal{F}_i[m]}{\partial m_j}$$

Final $$H\delta m = -g$$ ⇒

$$(J^T J + \beta)\delta m = -(J^T \delta d + \beta m)$$

$$\delta d = \mathcal{F}[m] - d^{obs}$$

General algorithm:



$$\text{minimize } \phi = \phi_d + \beta\phi_m$$

Initialize $m^{(0)}, \beta^{(0)}$

until convergence

$$H\delta m = -g$$

$$m^{(k+1)} = m^{(k)} + \alpha\delta m \text{ (line search)}$$

$$\beta^{(k+1)} = \frac{\beta^{(k)}}{\gamma} \quad \text{(cooling)}$$

Many variants:
- Solving system
- Cooling rate
- ….

Summary

$$\phi(m) = \frac{1}{2}\|\mathcal{F}[m] - d^{obs}\|^2 + \frac{\beta}{2}\|m\|^2$$

Linear	Non-linear
$d = Gm$	$d = \mathcal{F}[m]$
$(G^T G + \beta)\delta m = -(G^T d + \beta m)$	$(J^T J + \beta)\delta m = -(J^T \delta d + \beta m)$
	$\delta d = \mathcal{F}[m] - d^{obs}$
	$m^{(k+1)} = m^{(k)} + \alpha \delta m$

All understanding from linear problems is valid for nonlinear problems

DC resistivity

Governing PDE: electrostatic Maxwell's equations

- Faraday's law

$$\nabla \times \vec{e} = 0 \quad \rightarrow \quad \vec{e} = -\nabla \phi$$

- Ampere's law

$$\nabla \cdot \vec{j} = I\delta(r)$$

- Ohm's law

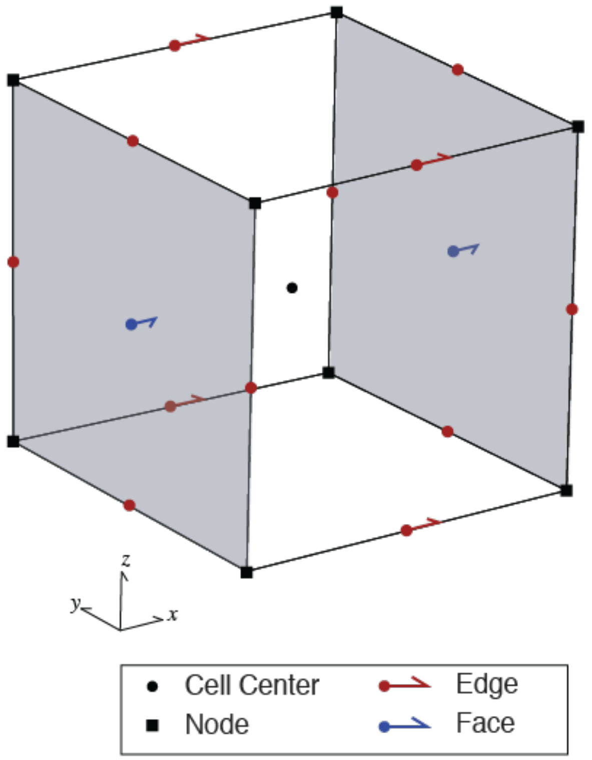
$$\vec{j} = \frac{1}{\rho}\vec{e}$$

Governing PDE

$$\nabla \cdot \frac{1}{\rho}\nabla \phi = -I\delta(r)$$

DC inversion

- Forward modelling
  - Nodal discretization for ϕ
  - Neumann boundary conditions $\frac{1}{\rho}\vec{e} = 0 \Big|_{\partial\Omega}$
  - Discrete equations $\underbrace{\mathbf{G}^\top \mathbf{M}^{\mathbf{e}}_{\sigma} \mathbf{G}}_{\mathbf{A}(\mathbf{m})} \phi = \mathbf{q}$



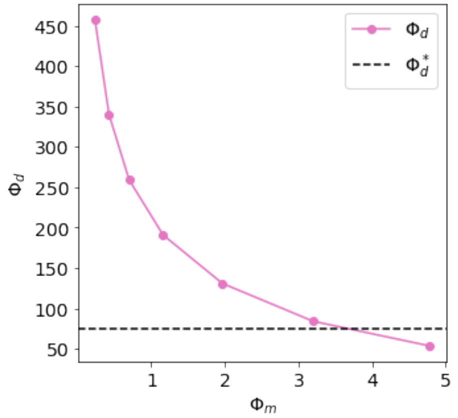
- Inversion
  - Invert for log resistivity (ensures positivity): $\mathbf{m} = \log(\rho)$

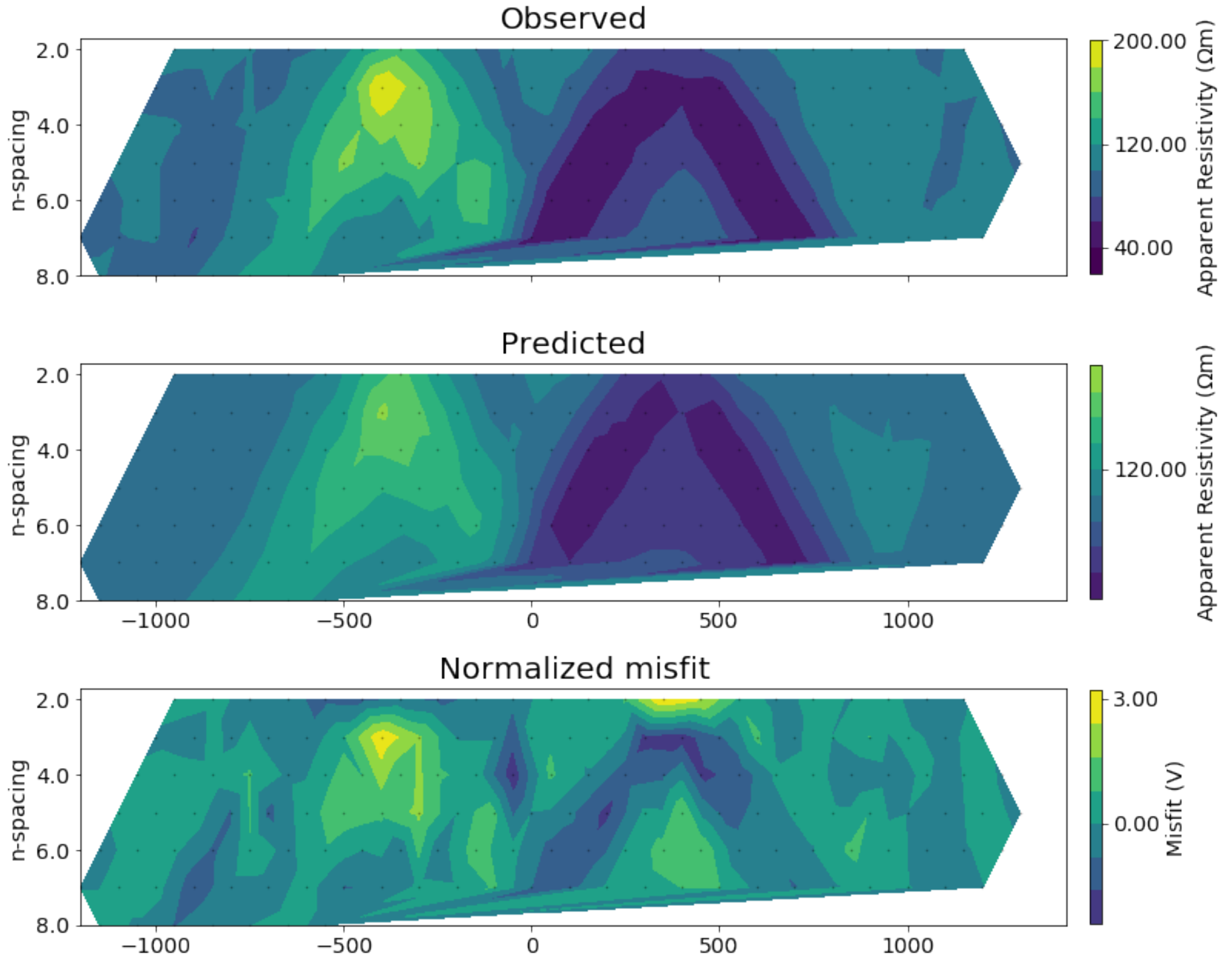
Tutorial: Cockett et al 2016, Pixels ad their neighbours

Evaluate results

Same as for linear problem

- Plot Tikhonov curve and some resulting models



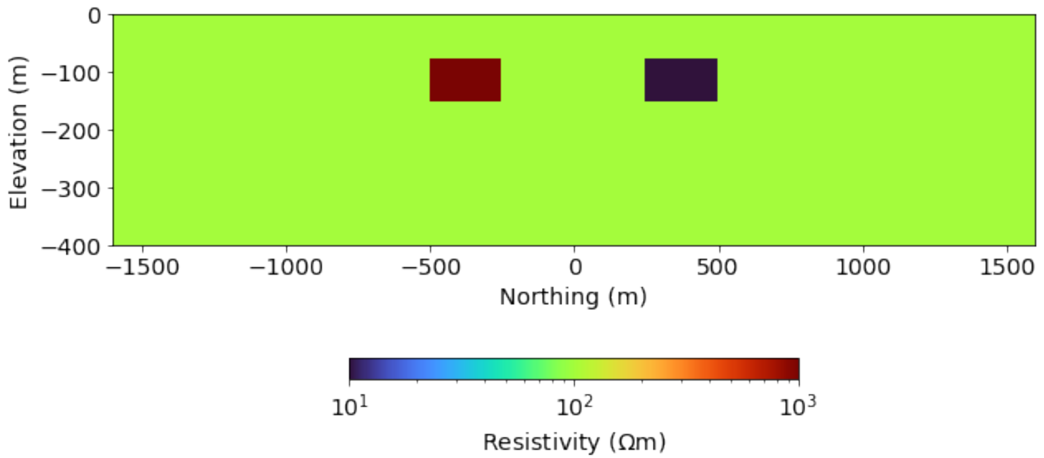


- Misfit

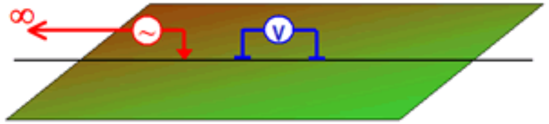
- Evaluate models

Example 2D DC resistivity

True resistivity model

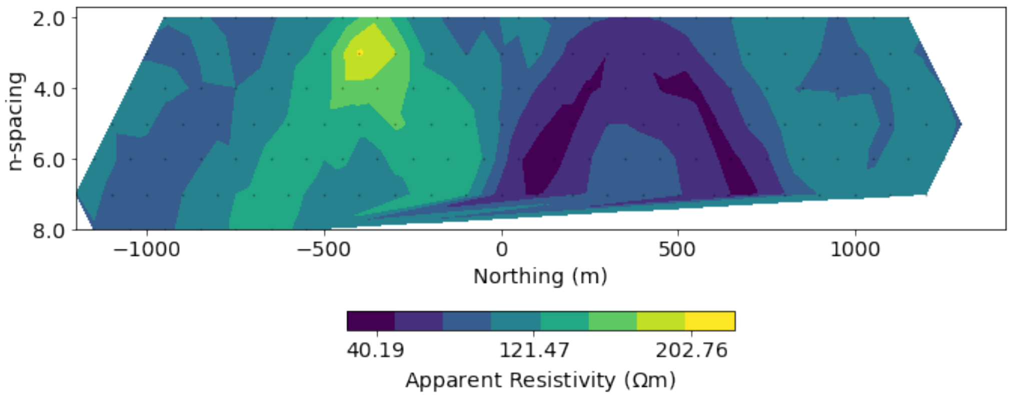


Pole-Dipole

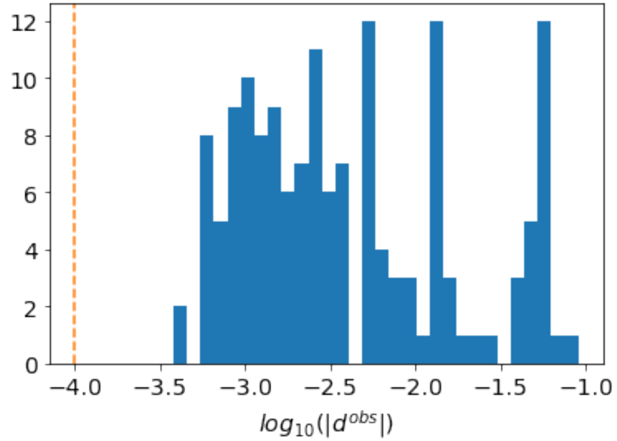


- Pole-dipole array
  - n-spacing = 8
  - Electrode-spacing = 100 m
  - # of data = 151
- 5% Gaussian noise added

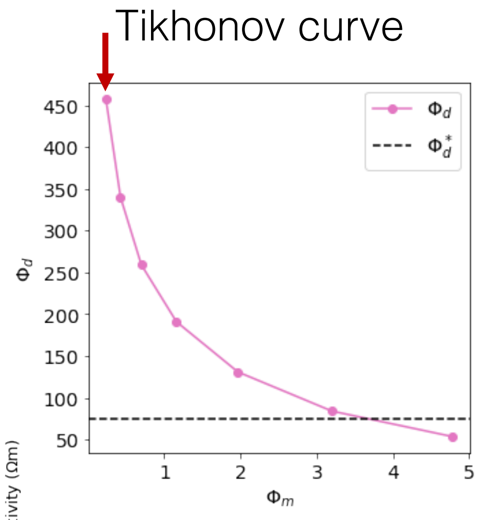
Apparent resistivity pseudo-section

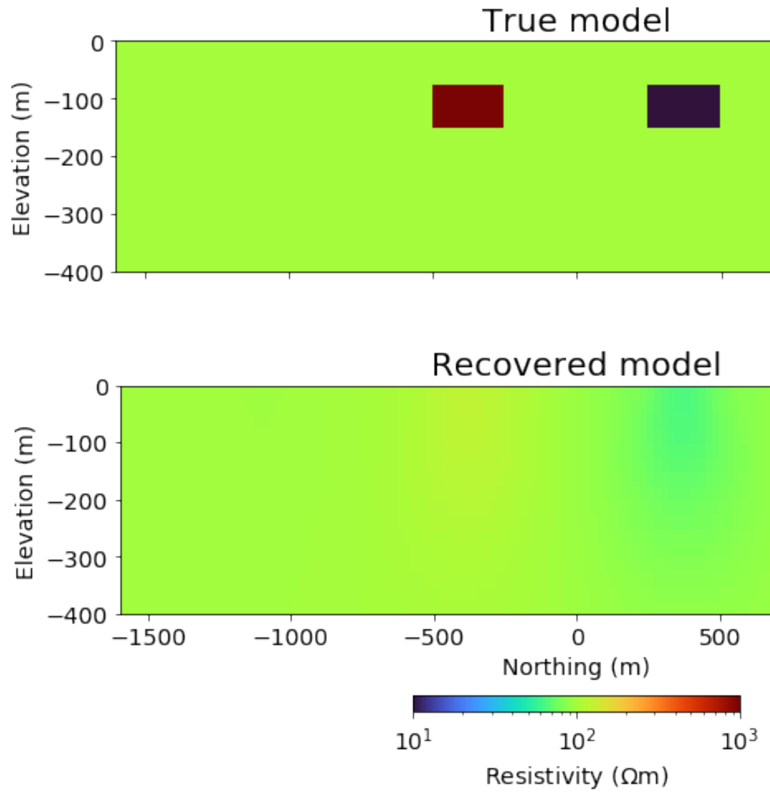


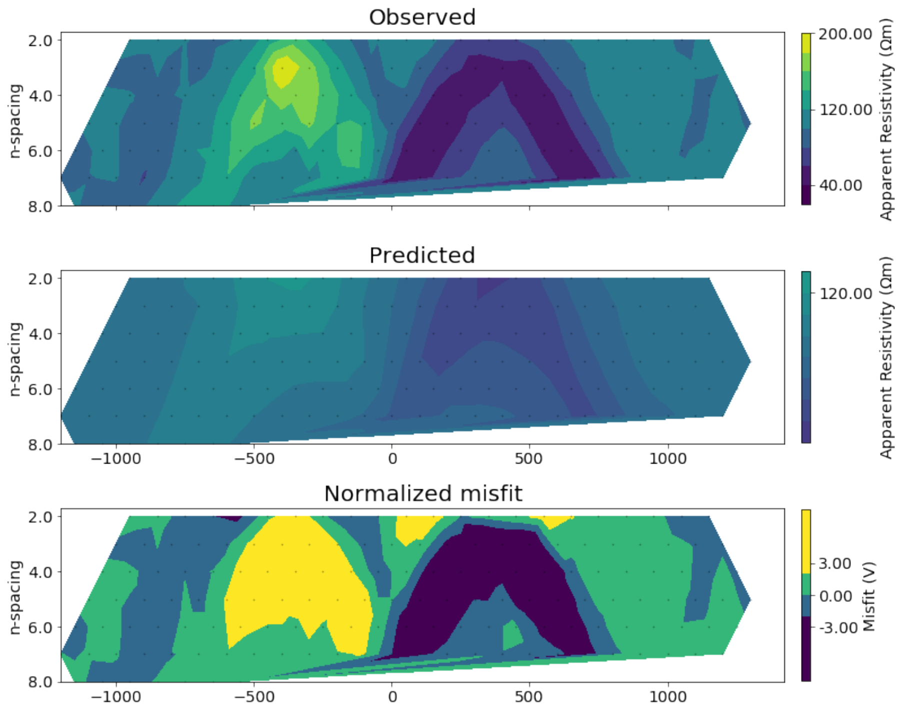
Log10 (Voltage)



DC inversion: iteration 1





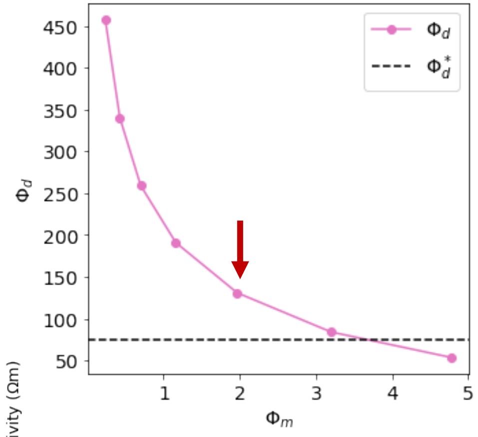


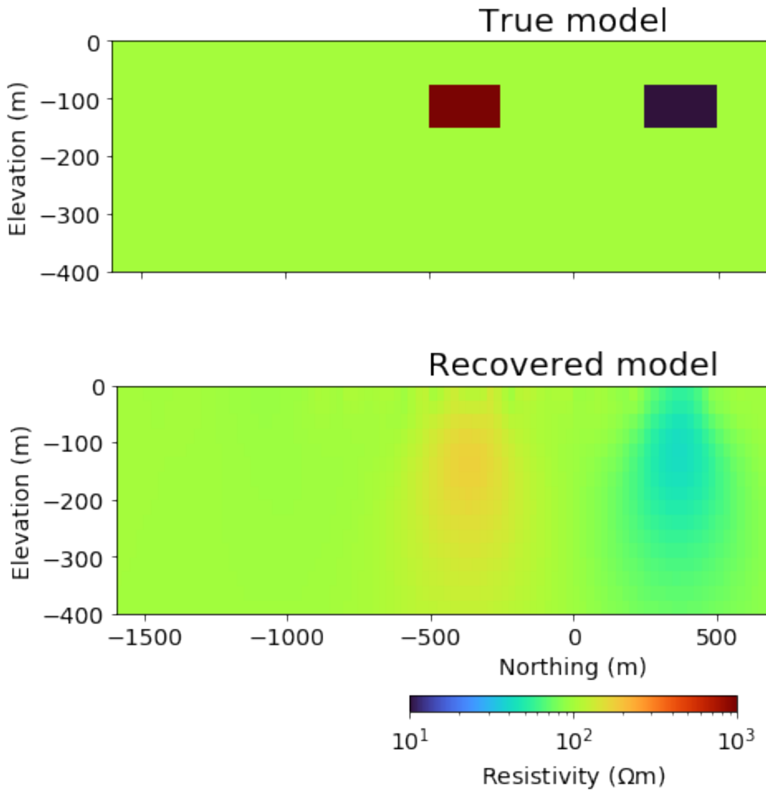
$\alpha_s = 0$

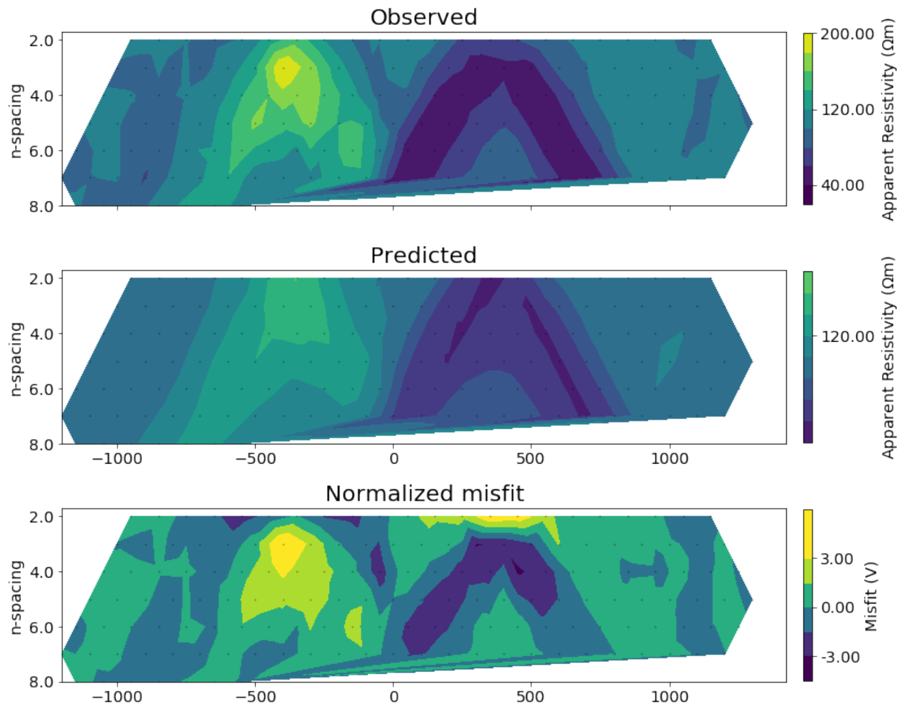
$\alpha_x = 1, \alpha_z = 1$

DC inversion: iteration 5

Tikhonov curve





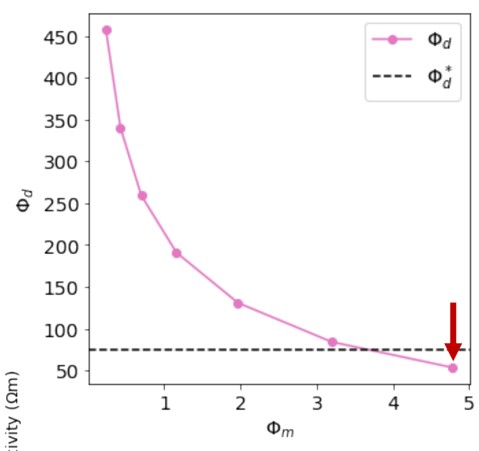


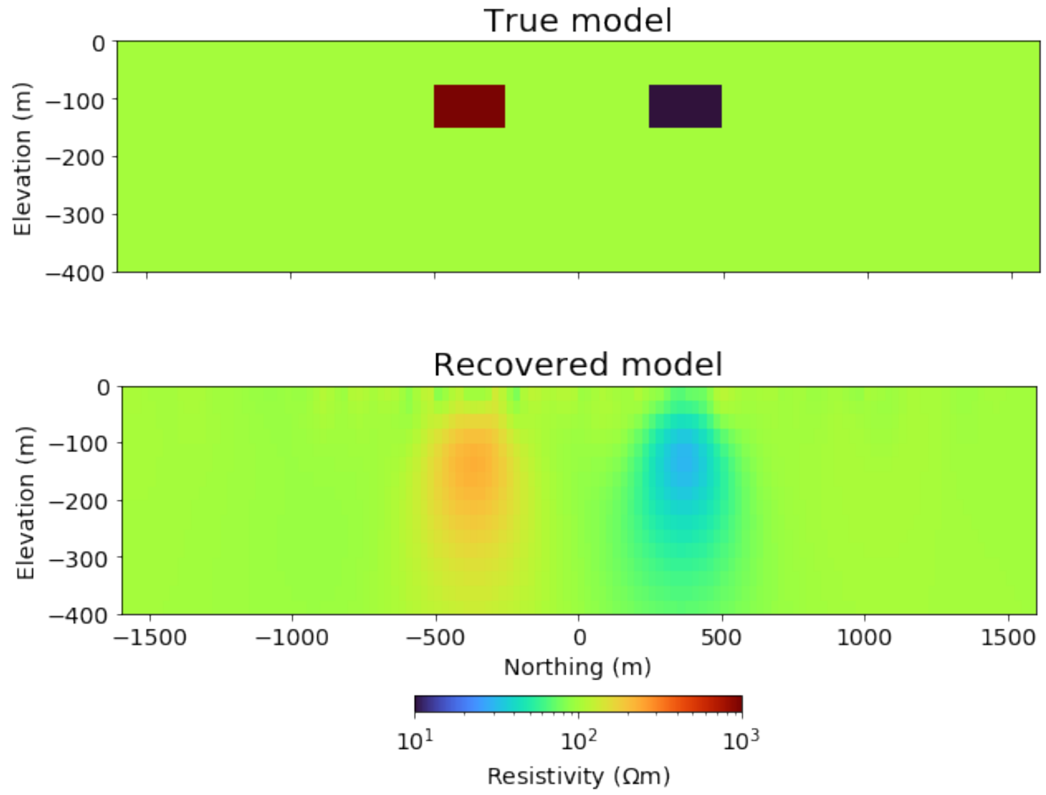
$\alpha_s = 0$

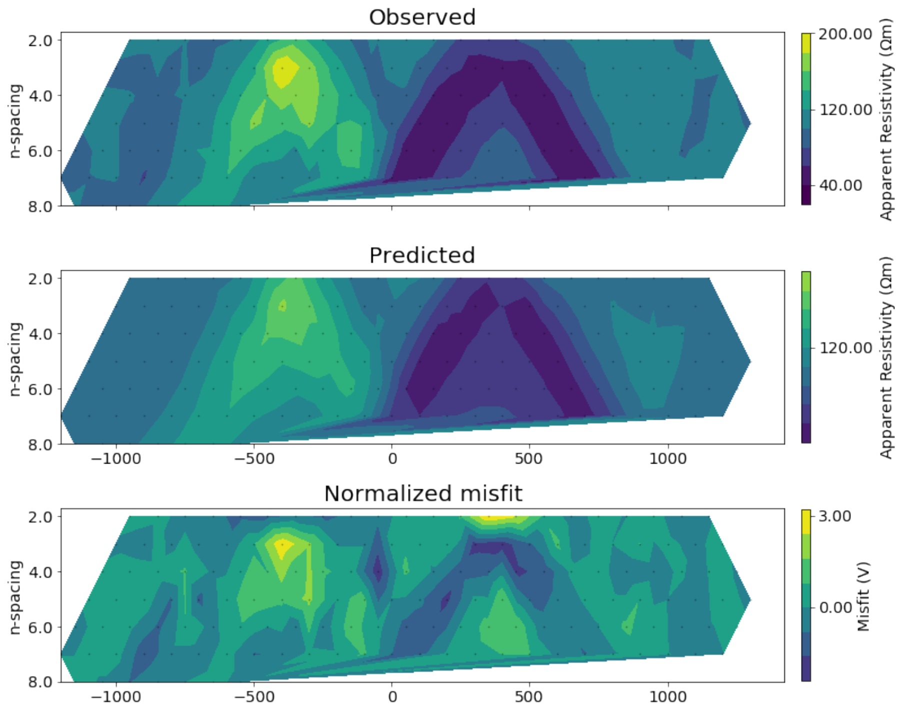
$\alpha_x = 1, \alpha_z = 1$

DC inversion: iteration 7

Tikhonov curve





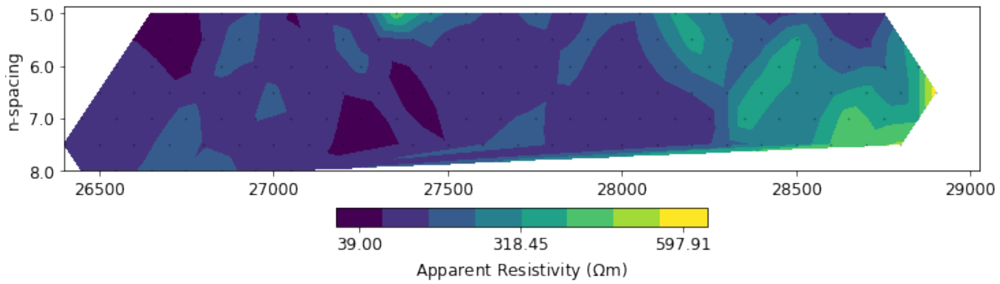


$\alpha_s = 0$

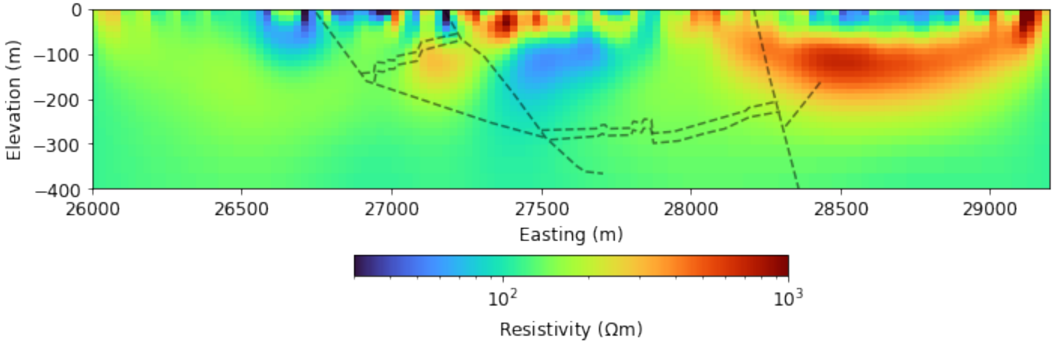
$\alpha_x = 1, \alpha_z = 1$

DC resistivity: Century

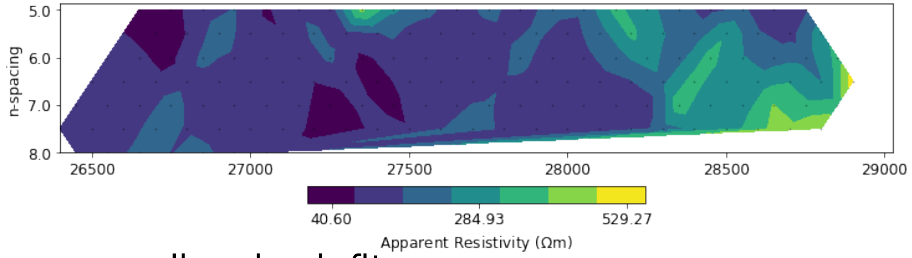
observed data



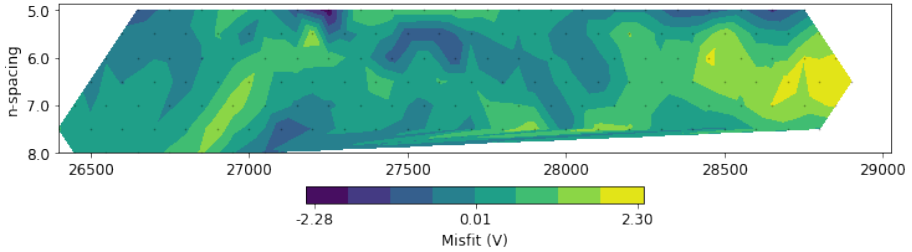
resistivity model



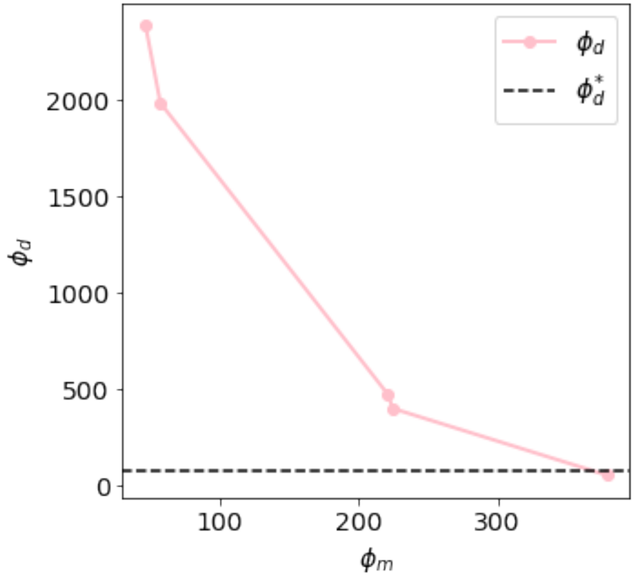
predicted data



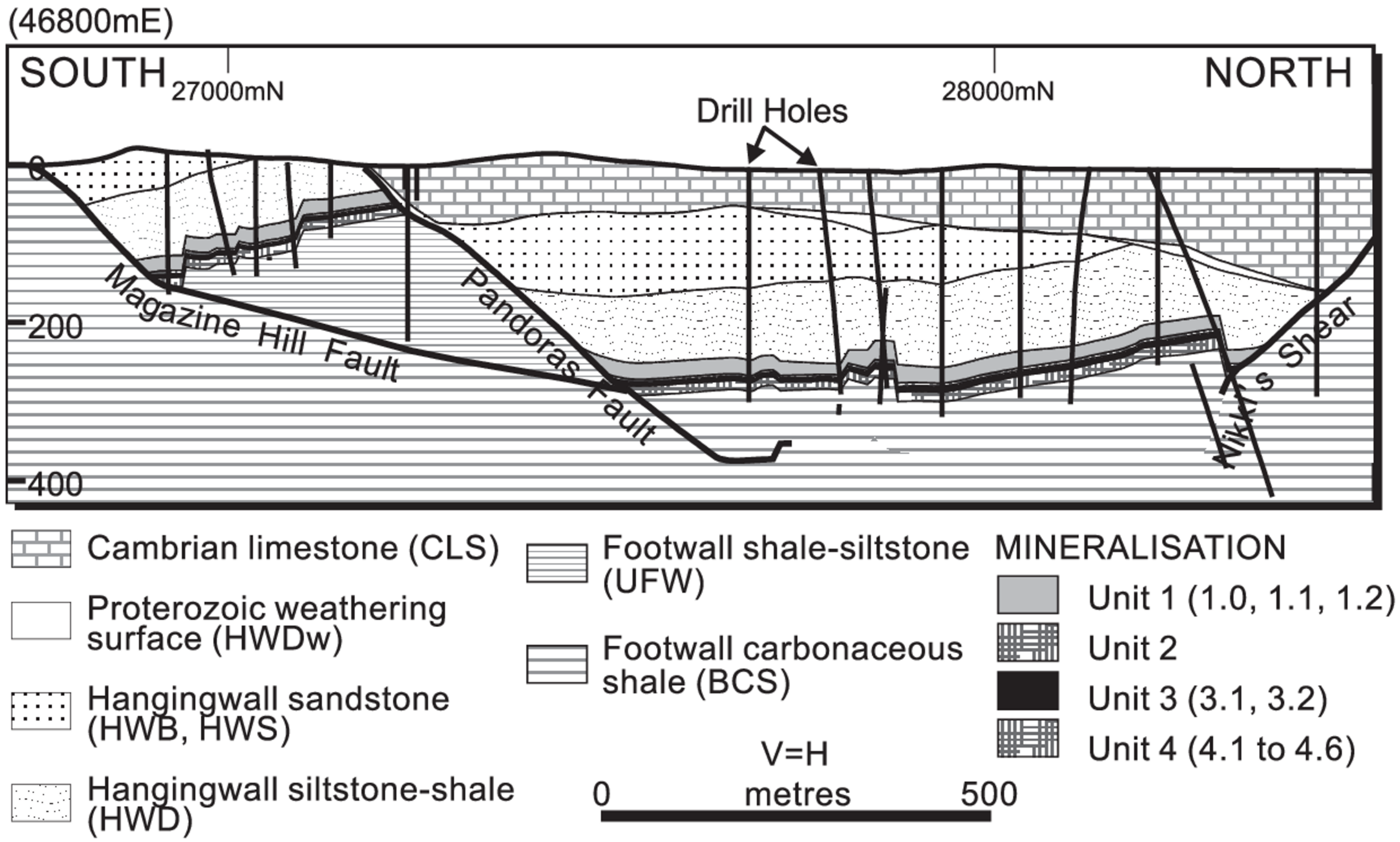
normalized misfit



Tikhonov curve

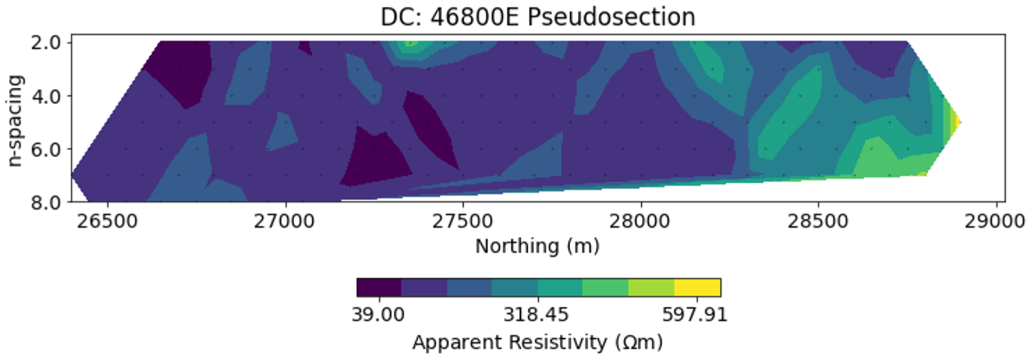


Century deposit

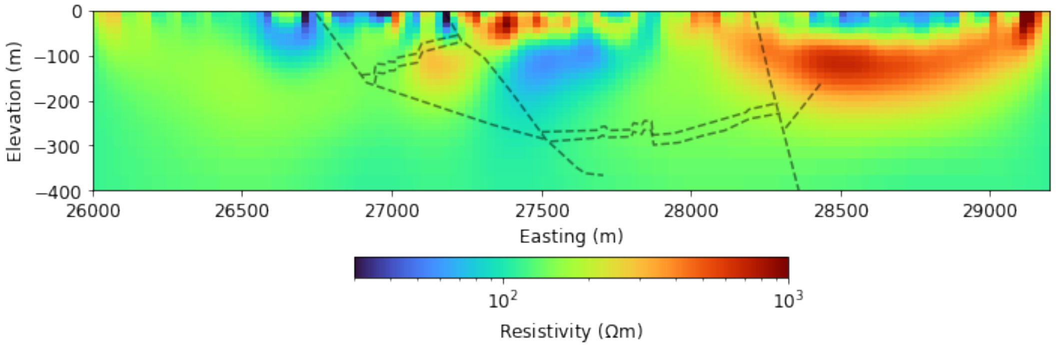


DC/IP Inversion is a 2-step process

Step 1: invert DC data

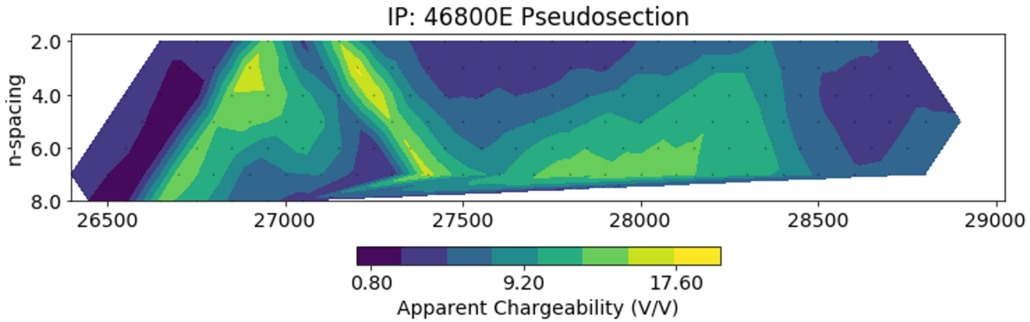


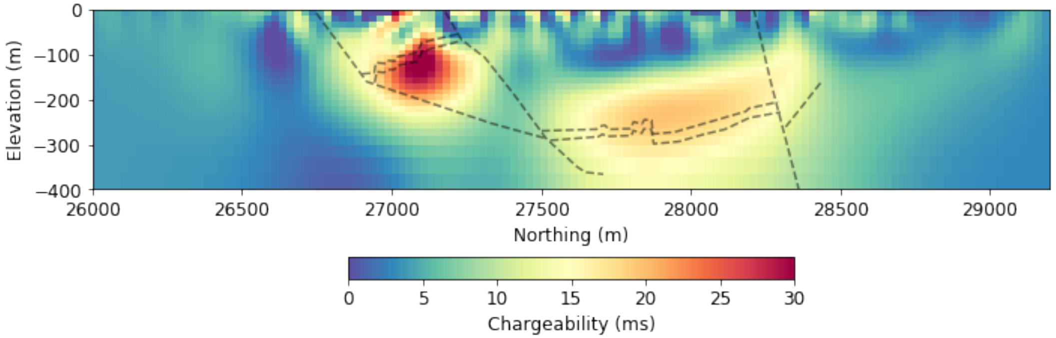
resistivity model



Step 2: invert IP data using sensitivities from recovered model in Step 1

chargeability model

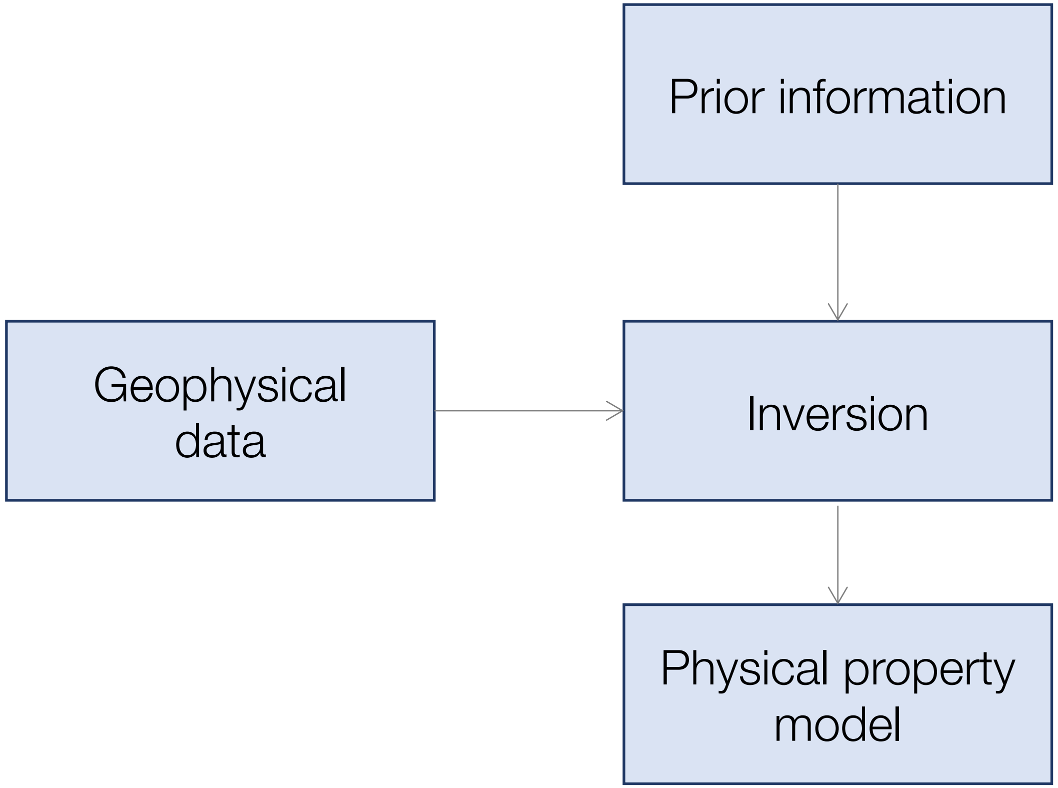




Constraining the inversion

What information is available?

- Geologic structure
- Geologic constraints
- Reference model
- Bounds
- Multiple data sets
- Physical property measurements



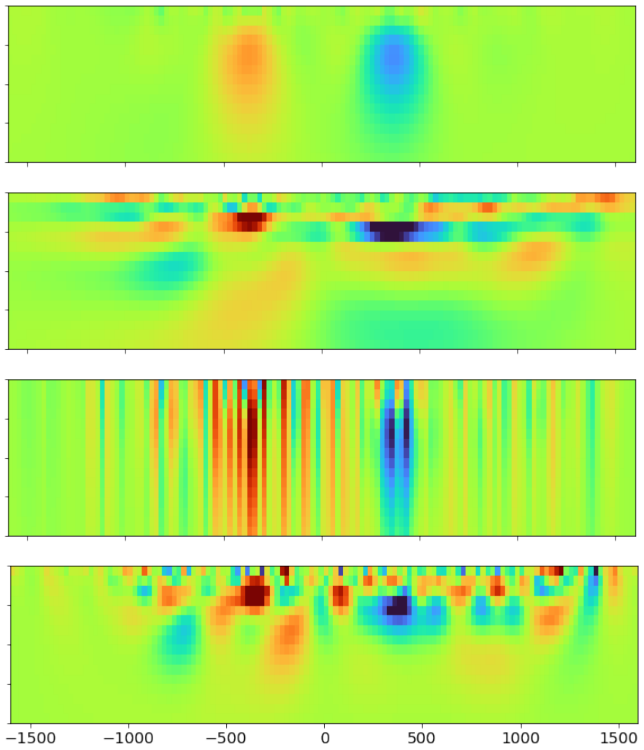
Constraining the inversion

Generic model norm

$$\phi_m = \alpha_s \int_v w_s \left(m - m_{\text{ref}}\right)^2 dv + \alpha_x \int_v w_x \left(\frac{d(m - m_{\text{ref}})}{dx}\right)^2 dx + \alpha_z \int_v w_z \left(\frac{d(m - m_{\text{ref}})}{dz}\right)^2 dz$$

Exploring the standard model norm

- Alpha weightings
- Weightings w's
- Reference model
- Combinations offer great flexibility



$\alpha_s = 0$
$\alpha_x = 1, \alpha_z = 1$

$\alpha_s = 0$
$\alpha_x = 1, \alpha_z = 0$

$\alpha_s = 0$
$\alpha_x = 0, \alpha_z = 1$

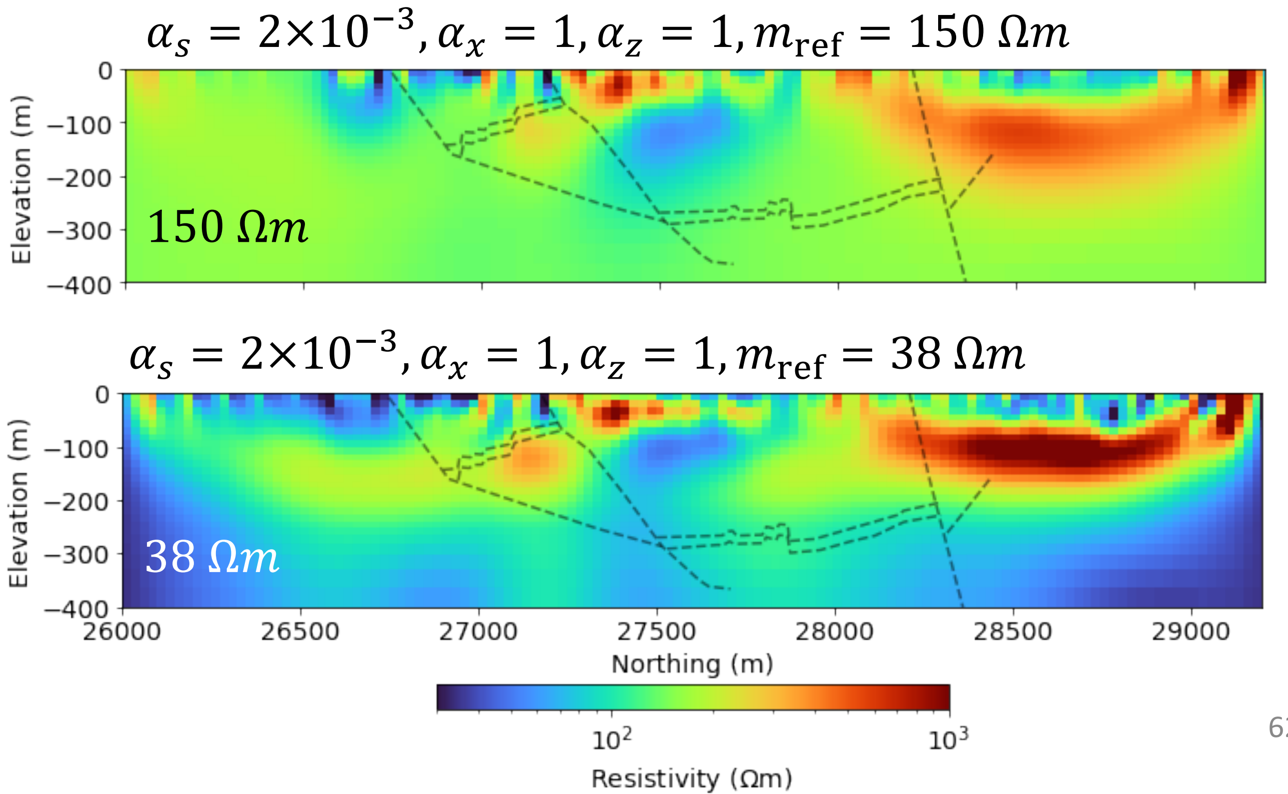
$\alpha_s = 1$
$\alpha_x = 0, \alpha_z = 0$

Reference model and its uses

Generic model norm

$$\phi_m = \alpha_s \int_v w_s \left(m - m_{\text{ref}}\right)^2 dv + \alpha_x \int_v w_x \left(\frac{d(m-m_{\text{ref}})}{dx}\right)^2 dx + \alpha_z \int_v w_z \left(\frac{d(m-m_{\text{ref}})}{dz}\right)^2 dz$$

- Simple or complex
- Used in derivative terms or not
- w's used to attach confidence in the reference model
- Can be used to
  - incorporate additional information
  - Hypothesis testing
  - Depth of investigation for survey



Use of a reference model for depth of investigation

Background to DOI

$$\text{doi index} = \frac{m^1 - m^2}{m^1_{ref} - m^2_{ref}}$$

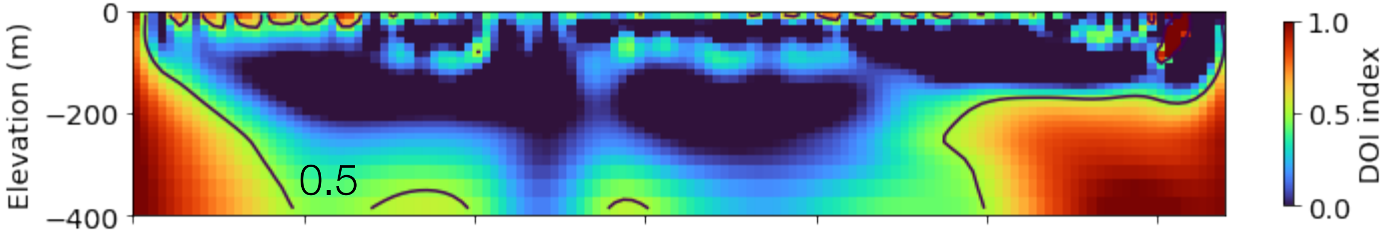
m_1: recovered model with m^1_{ref}

m_2: recovered model with m^2_{ref}

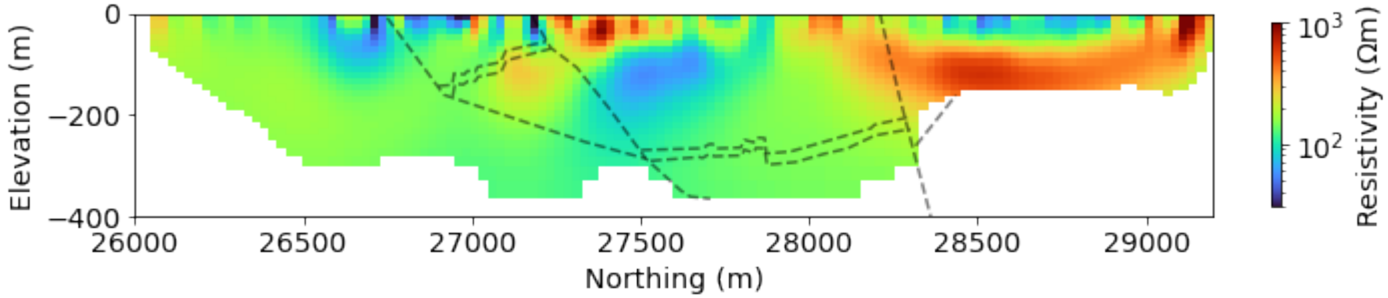
(Oldenburg and Li, 1999)

Example from Century deposit

DOI index



Resistivity model above the DOI (DOI index=0.5)



Use weighting functions

model norm: $\phi_m = \alpha_s \int_v \underline{w_s} \left(m - m_{\text{ref}} \right)^2 dv + \alpha_x \int_v \underline{w_x} \left(\frac{d(m - m_{\text{ref}})}{dx} \right)^2 dx + \alpha_z \int_v \underline{w_z} \left(\frac{d(m - m_{\text{ref}})}{dz} \right)^2 dz$

- Incorporate confidence in model or derivative
- Hypothesis testing
- Used to incorporate sensitivity weighting
- Important for potential fields
- Generate more realistic models

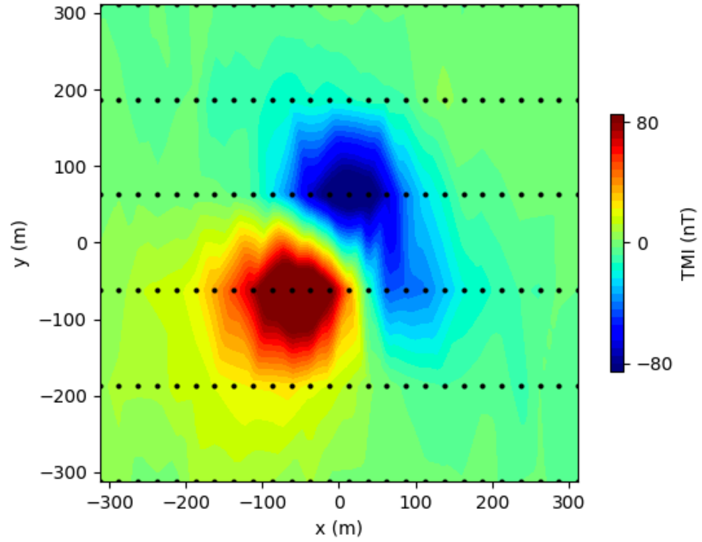
Sensitivity weighting

Consider $Gm = d$ and minimize $||m||^2$

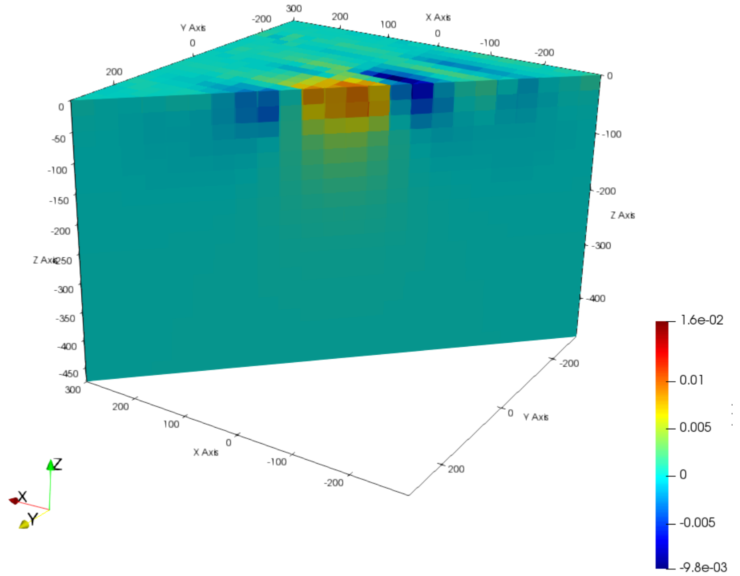
Easiest way to generate signal is to locate m where G is large.

In magnetics this produces a module with susceptibility at the surface

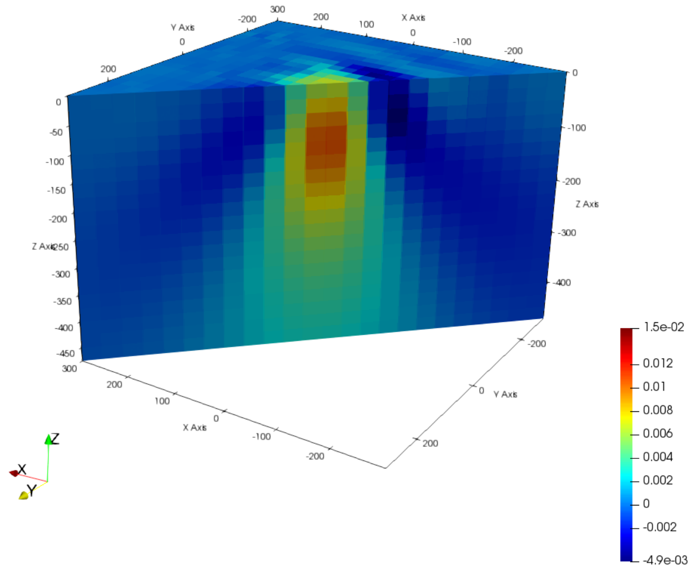
Magnetic data



Recovered susceptibility



Sensitivity weighted susceptibility

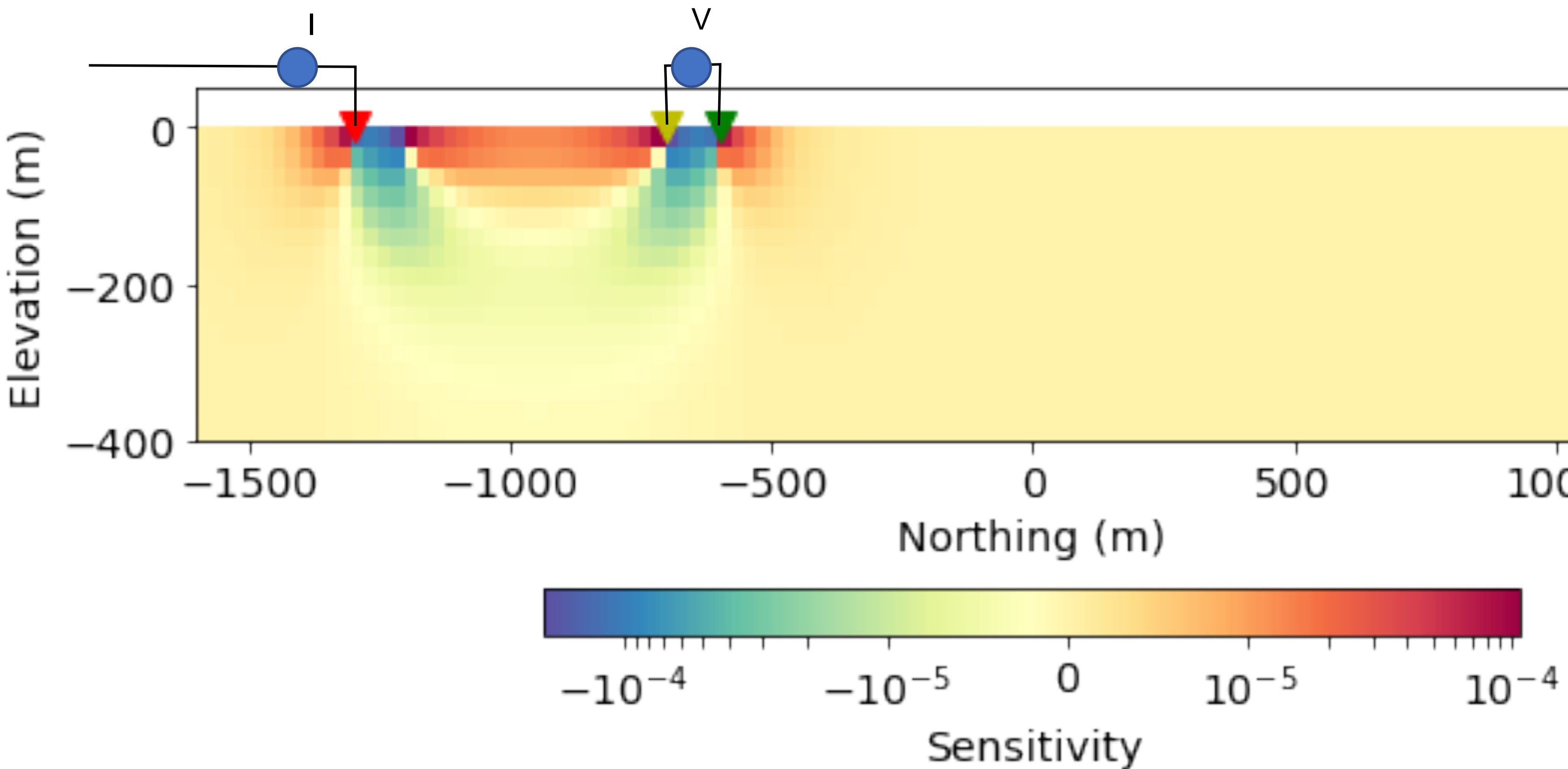


Sensitivity weighting

Consider $Gm = d$ and minimize $||m||^2$

Easiest way to generate signal is to locate m where G is large.

Sensitivity in a DC experiment



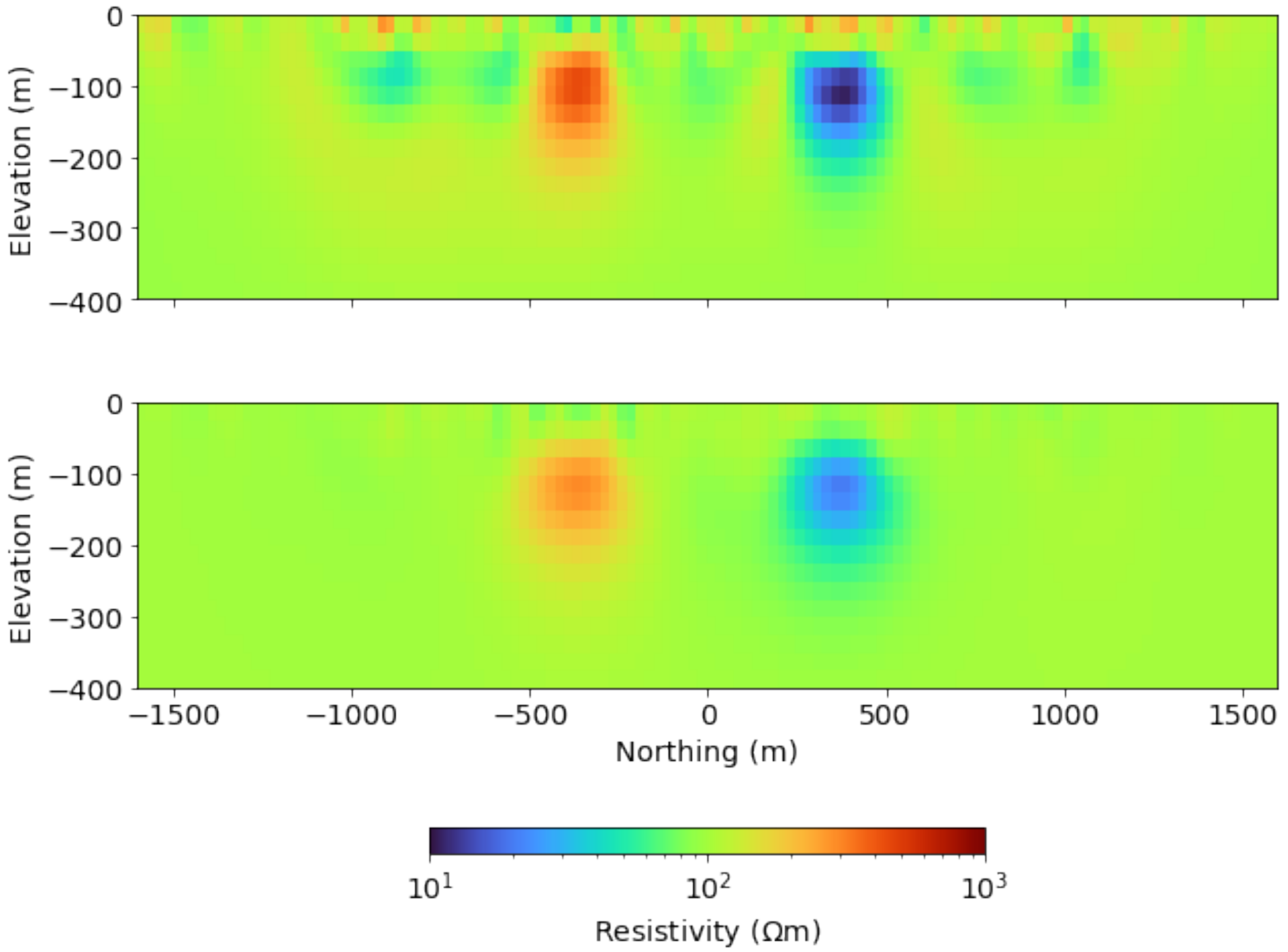
Sensitivity weighting for DC

$$w_j = \Sigma_{i=1}^{N} \sqrt{J_{ij}^2}$$

i-th datum
j-th model

Without sensitivity weighting

With sensitivity weighting



Bound Constraints

- Physical property bounds in each cell

$$\mathbf{m}_L \leq \mathbf{m} \leq \mathbf{m}_U$$

- Projected Gradient Gauss-Newton (Kelly, 1999; Haber, 2015)
  - At each GN iteration

$$\delta \mathbf{m} = \mathbf{H}^{-1} \delta \mathbf{d} + \alpha \mathbf{g}$$

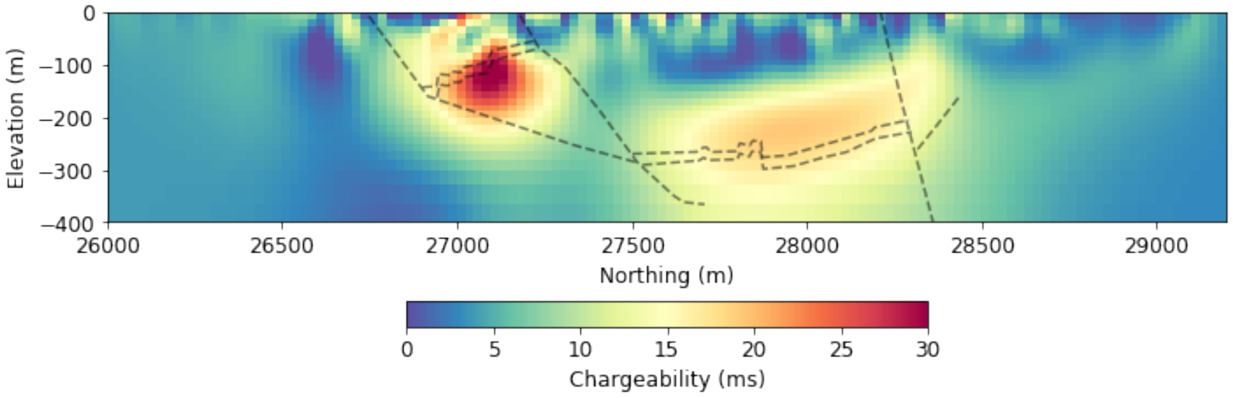
$\mathbf{H}$: Hessian for cells not at the bounds
$\mathbf{g}$: gradient for cells at the bounds
α: scalar

Positivity $\mathbf{m} \geq 0$

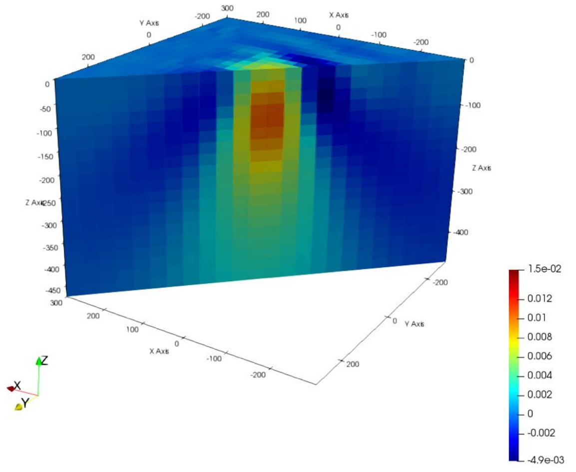
Enforcing positivity

Chargeability model	Susceptibility model

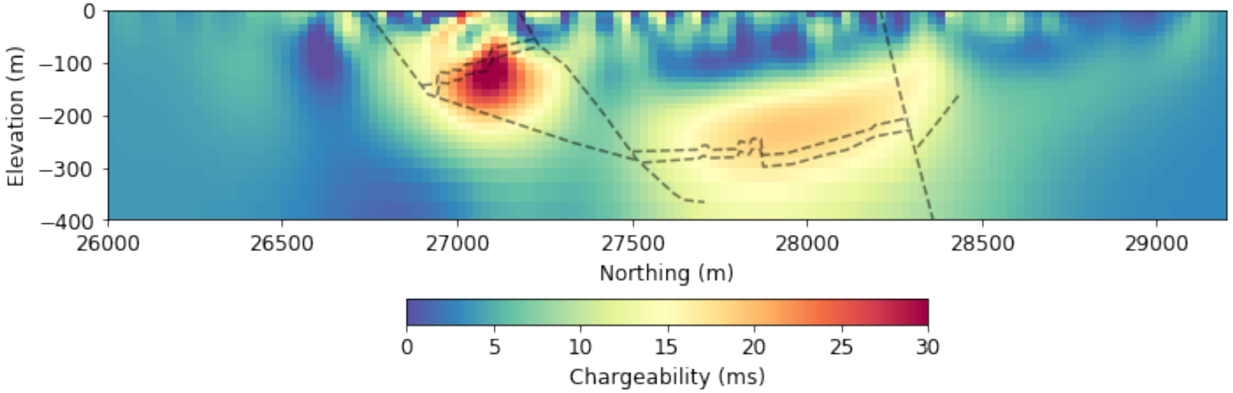
without positivity



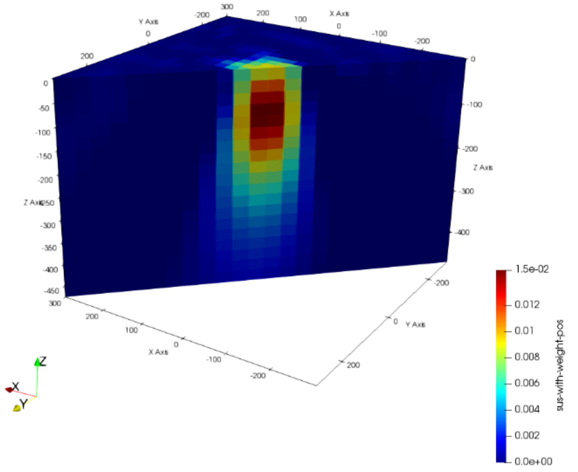
without positivity



with positivity

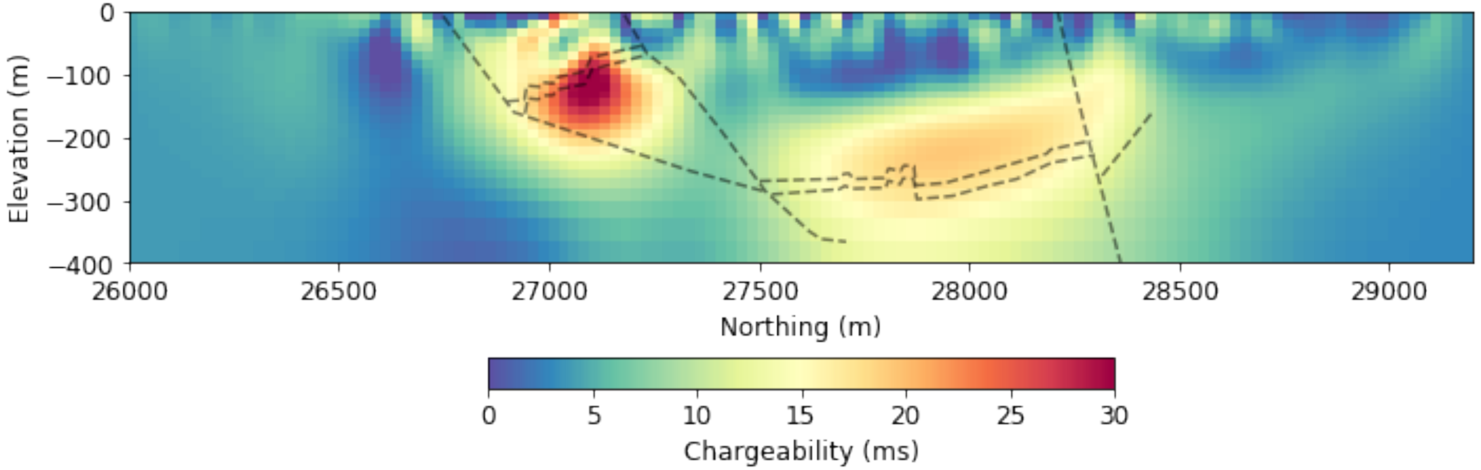


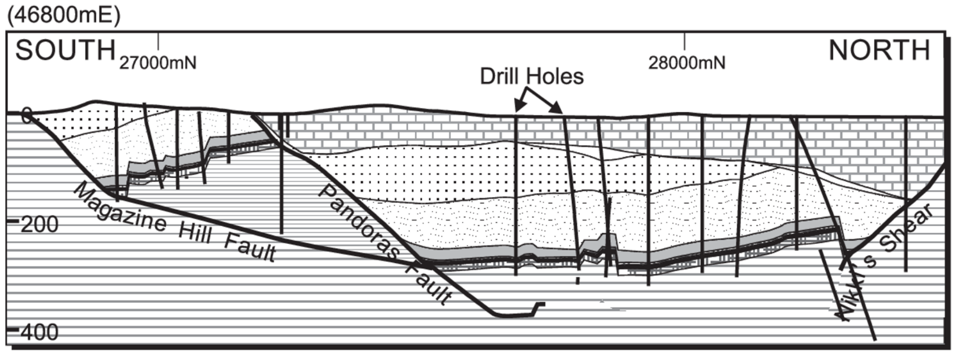
with positivity



Structural information

- Body is at about the right depth but it is still smoothed out
- Want a solution that produces a thin mineralized zone
- Makes the faults more distinct





We can do this by altering the model norm

$$\phi_m = \alpha_s \int_V w_s |m - m_{\text{ref}}|^{p_s} dV + \alpha_x \int_V w_x \left| \frac{d(m - m_{\text{ref}})}{dx} \right|^{p_x} dV + \alpha_z \int_V \left| \frac{d(m - m_{\text{ref}})}{dz} \right|^{p_z} dV$$

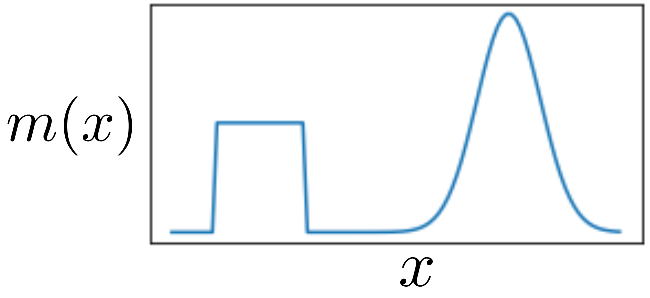
Why Lp?

- Work so far we have used L_2 norms

$$\phi_m = \int m^2(x)dx \quad \xrightarrow{\text{Discretize}} \quad \phi_m = \sum_{i=1}^{M} m_i^2 v_i$$

- General L_p-norm

$$\phi_m = \sum_{i=1}^{M} |m_i|^p v_i \qquad 0 \le p \le 2$$



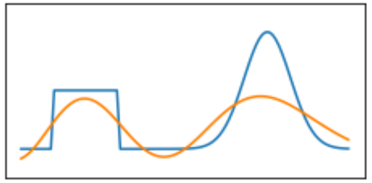
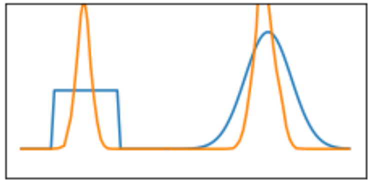
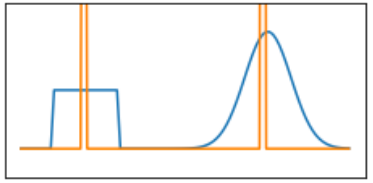
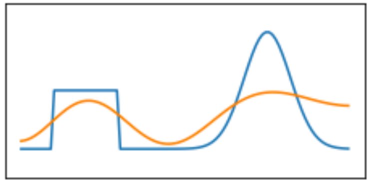
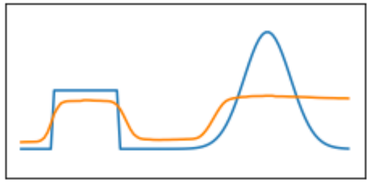
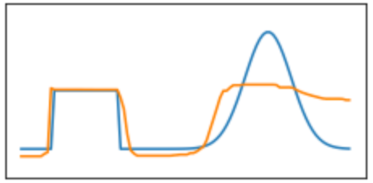
ϕ_m	p=2	p=1	p=0.5	p=0
	69	55	54	100

General character

$$\phi_m = \sum_{i=1}^{M} |m_i|^p v_i$$

- Geometric character
  - p=2: all elements close to zero
  - p=1: sparse solution, # of non-zero elements are ≤ # of data
  - p=0: minimum support, model with the fewest number of elements

- 1D problem

	p=2	p=1	p=0
only ϕ_s			
only ϕ_x			

General Lp objective function

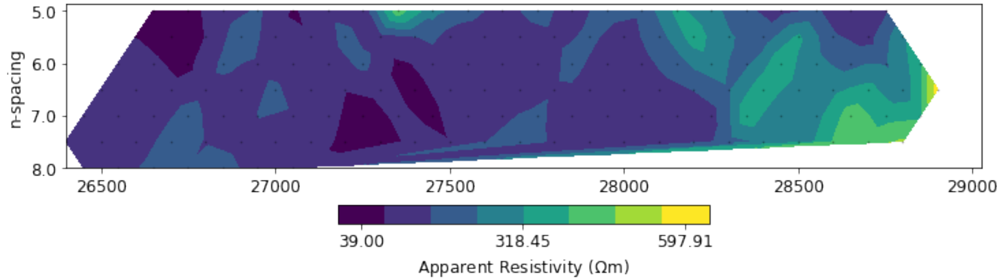
Each component of a 3D objective function can have its own Lp-norm

$$\phi_m = \alpha_s \int_V w_s |m - m_{\text{ref}}|^{p_s} dV + \alpha_x \int_V w_x \left| \frac{d(m - m_{\text{ref}})}{dx} \right|^{p_x} dV + \alpha_z \int_V \left| \frac{d(m - m_{\text{ref}})}{dz} \right|^{p_z} dV$$

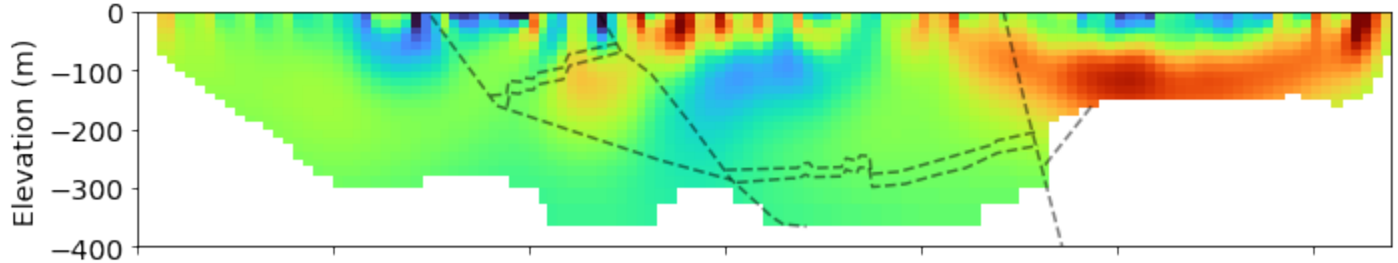
$$0 \leq p_j \leq 2$$

Lp inversion of DC data

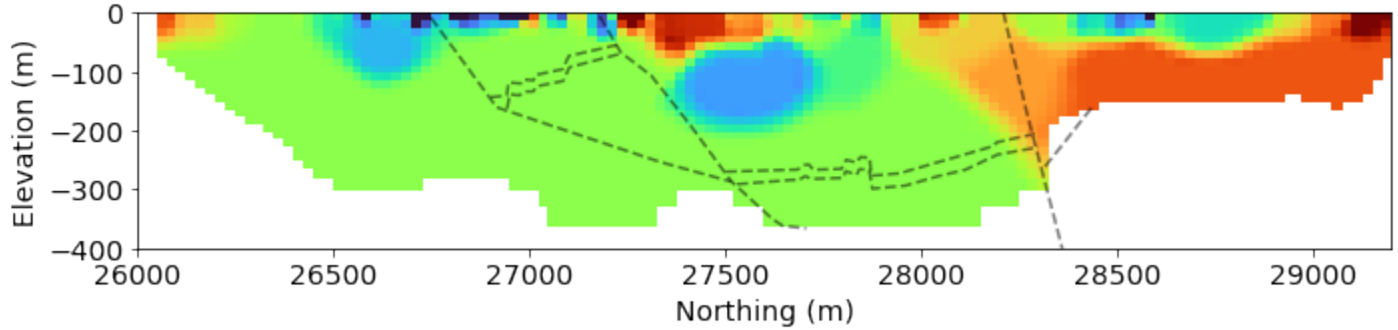
observed data



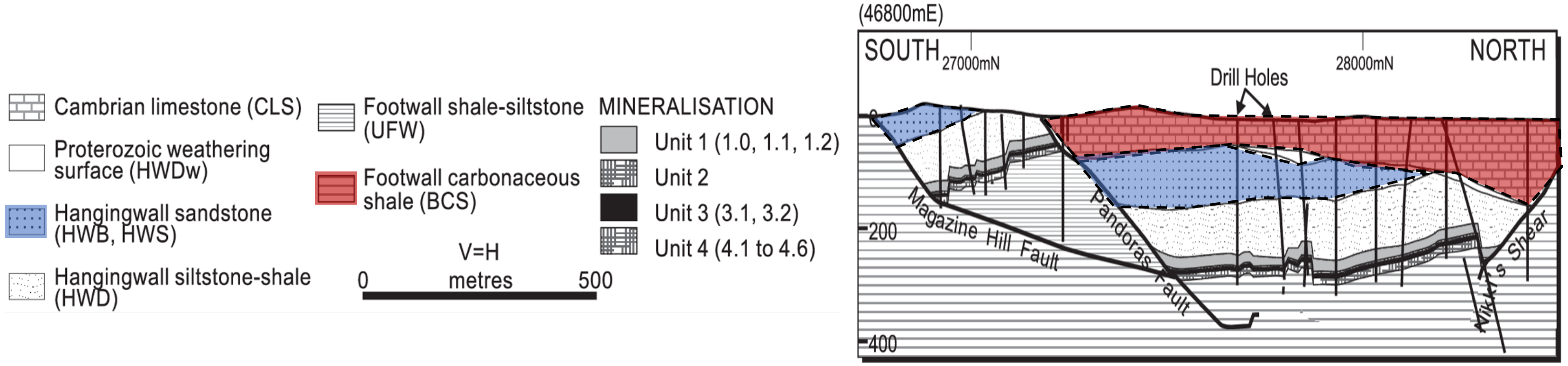
Resistivity $p_s = p_x = p_z = 2$



$p_s = p_x = p_z = 0$

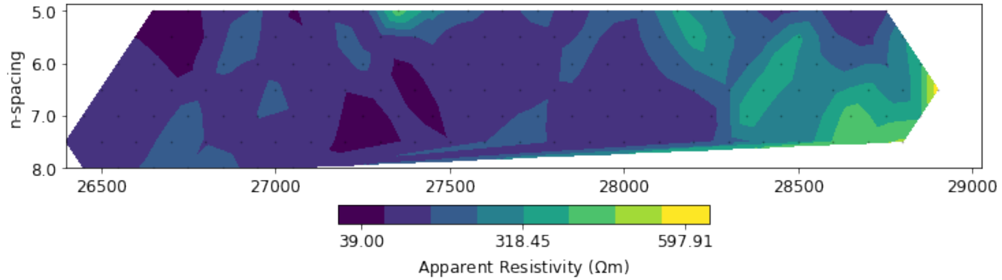


[Fournier and Oldenburg, 2019](#)



Lp inversion of DC data

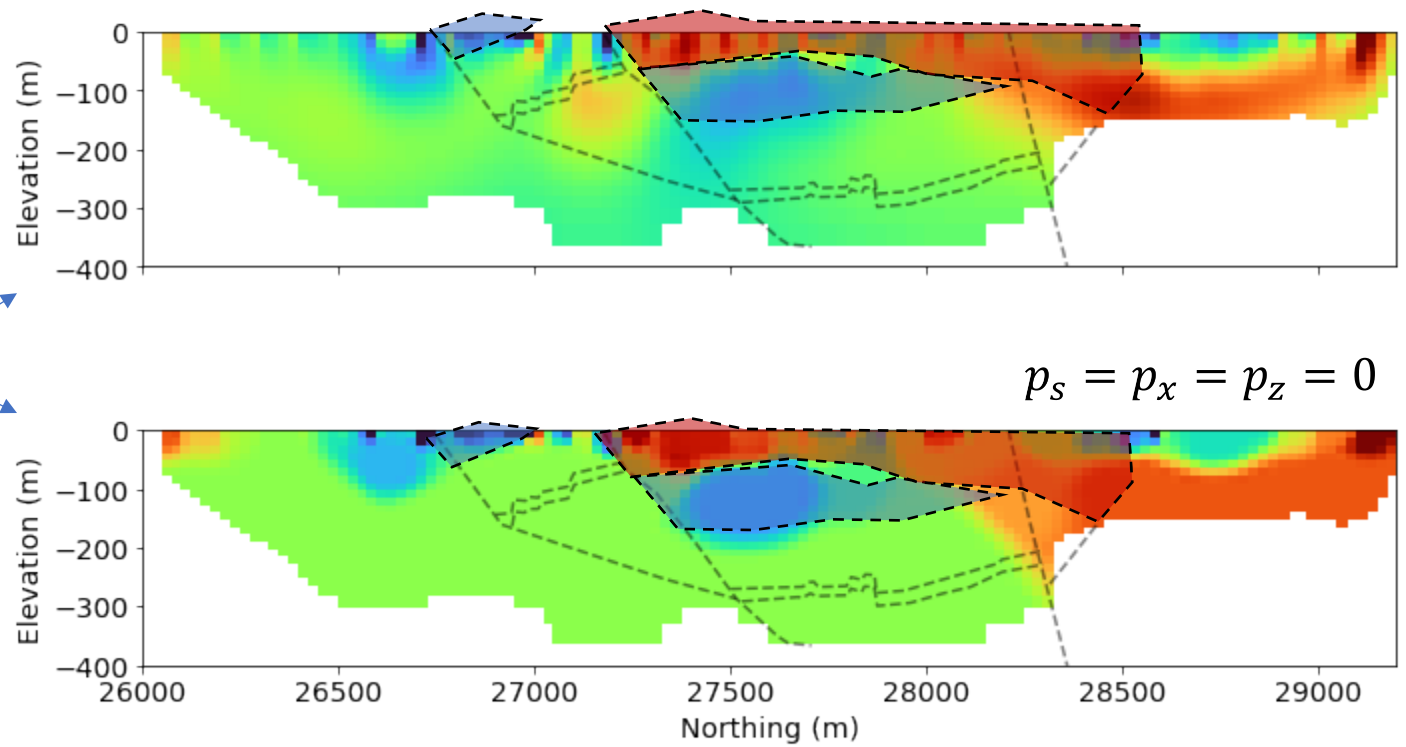
observed data



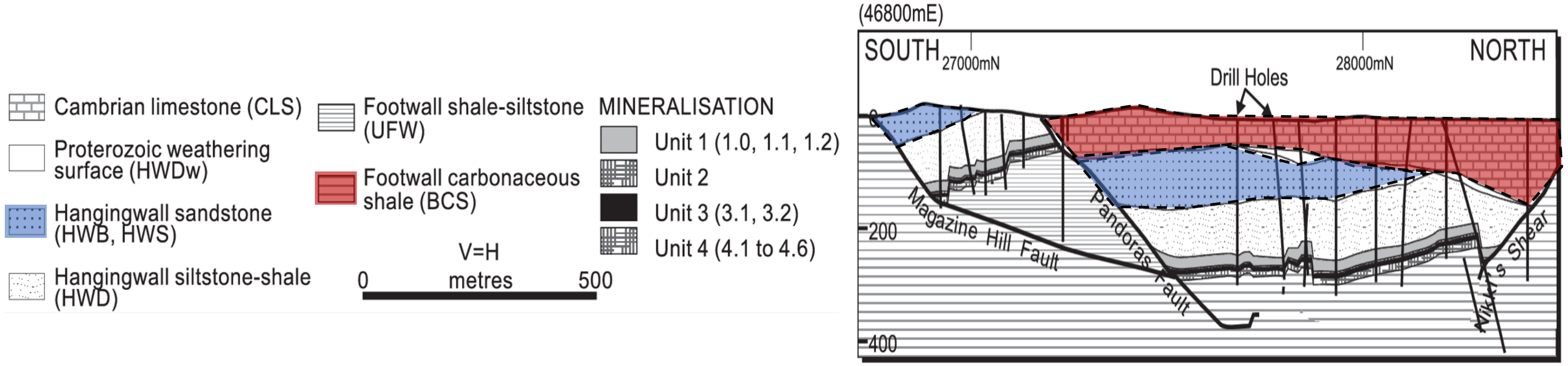
Resistivity

$p_s = p_x = p_z = 2$

$p_s = p_x = p_z = 0$



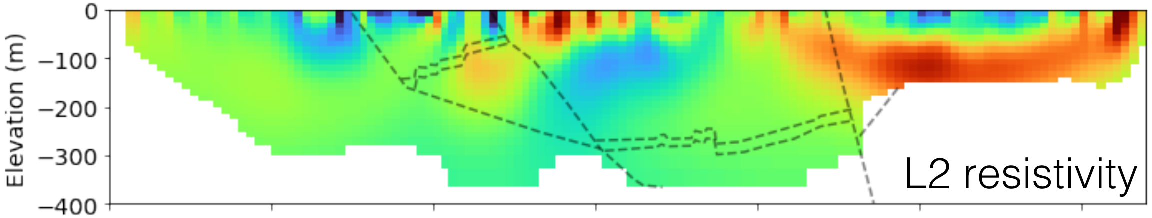
[Fournier and Oldenburg, 2019]

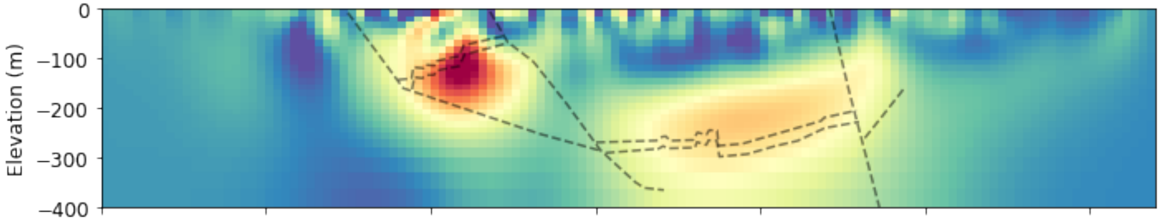


Lp inversion of IP data

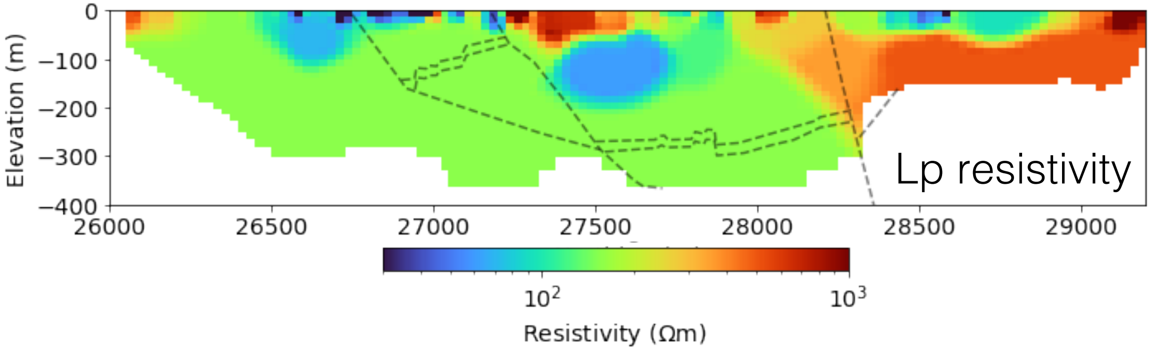
Resistivity model (L2) | Chargeability models

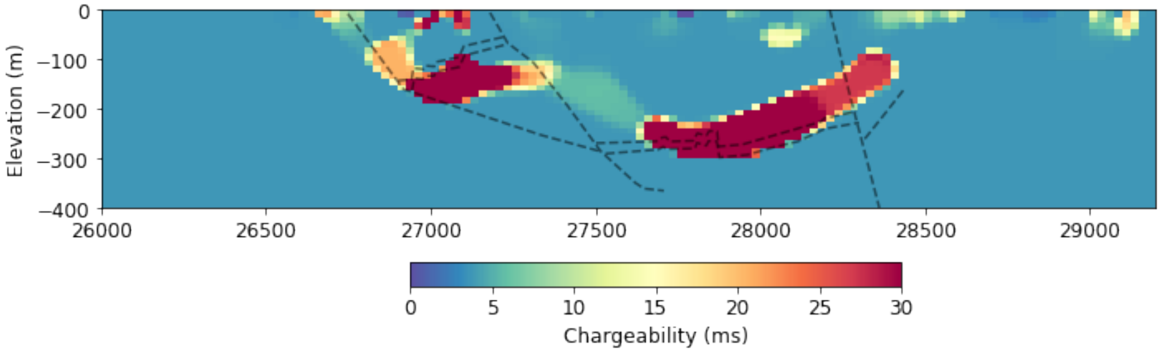
$$p_s = p_x = p_z = 2$$

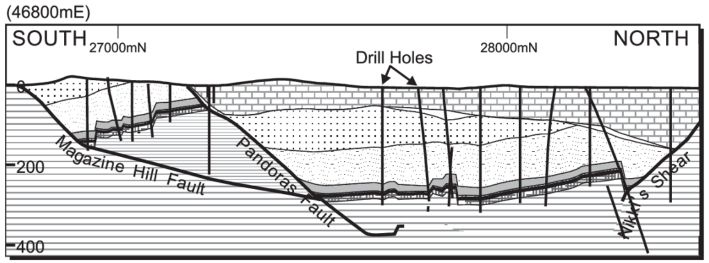




$$p_s = p_x = p_z = 0$$







[Fournier and Oldenburg, 2019]

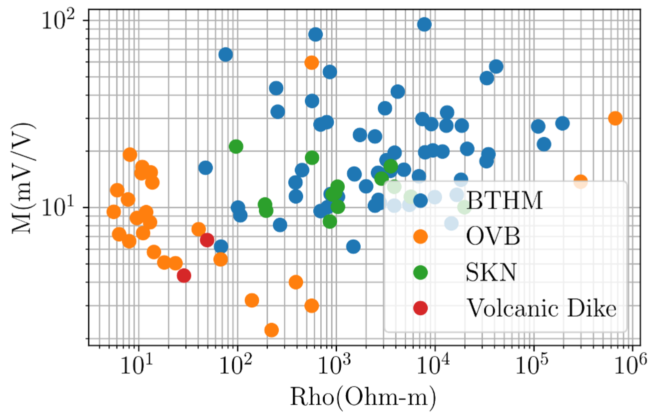
What other information is available?

Petrophysics: each rock units each with range of physical properties

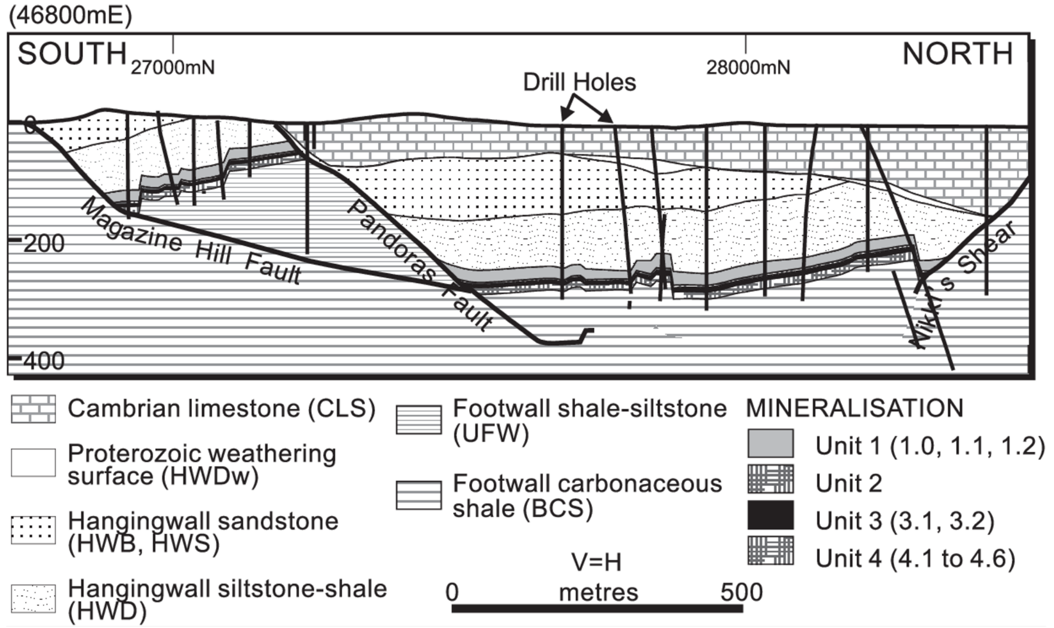
Geology: Lithology from drill holes

- Petrophysics
- Well-logs

Multiple properties

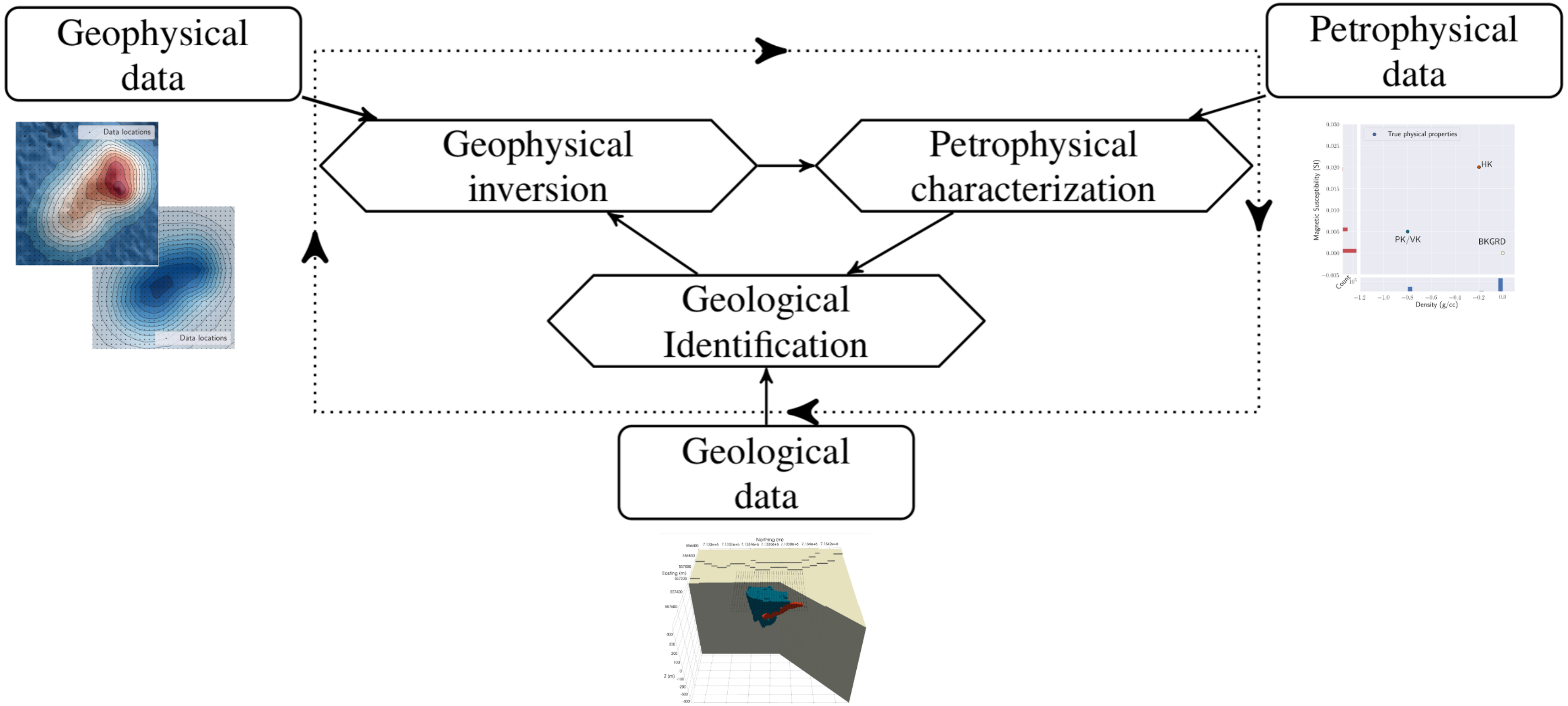


Drill cores



Linking Geophysics, Petrophysics and Geology

[Astic and Oldenburg, 2020]



Petrophysical characterization and geological identification are encoded in model norm.

$$\Phi_s(\mathbf{m}) = \frac{1}{2}\sum_{i=1}^{n} ||W_s(\Theta, z_i^*)(\mathbf{m}_i - \mathbf{m}_{ref}(\Theta, z_i^*))||_2^2$$

Thank you!

- SimPEG: https://simpeg.xyz/
- Inversion resources: curvenote.com/@geosci/inversion-module