Blocky models in minimum-structure inversions

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Summary

The inclusion of diagonal spatial differences in an l_1 -type measure of model structure enables piecewise-constant, blocky models with dipping interfaces to be constructed by minimum-structure inversion procedures. Examples are given for 2D MT inversion.

Introduction

Motivation

Minimum-structure inversion procedures (that is, ones in which a measure of the amount of structure in the model is minimized in conjunction with a measure of data misfit, and in which the parameters being sought are the physical properties in the cells in an otherwise fixed mesh) are generally robust and reliable. However, the features in the models which are produced are typically smearedout and fuzzy. The aim of the work presented here is to develop a minimumstructure inversion procedure that retains the robustness and reliability of existing implementations, but which can generate models comprising relatively uniform regions separated by sharp, distinct interfaces.

Theory (continued)

General measures

The general form of the measures ϕ_d and ϕ_k is:

$$\phi(\mathbf{x}) = \sum_{j=1}^{N} \rho(x_j).$$

There are numerous possibilities: (i) the classic l_2 norm with

$$\rho(x) = x^2,$$

(*ii*) the l_p norm with

$$\rho(x) = |x|^p,$$

(*iii*) the *M*-measure of Huber for which

$$\rho(x) = \begin{cases} x^2 & |x| \le c, \\ 2c|x| - c^2 & |x| > c, \end{cases}$$

and (*iv*) the perturbed l_p norm-like measure of Ekblom for which

Example

A synthetic MT data-set was generated from the conductivity model shown in Figure 2 – real and imaginary parts of the E-polarization impedances at 17 locations at three frequencies (3, 10 and 30 Hz). Gaussian random noise of standard deviation equal to 1% of a datum was added.





Previous work

A number of authors have successfully modified the traditional minimumstructure inversion procedure to use non- l_2 -norm, non-sum-of-squares measures and thus produce models with sharp interfaces: Farquharson & Oldenburg (1998) for 1D EM; Portniaguine & Zhdanov (1999) for 3D gravity; Loke et al. (2003) and Farquharson & Oldenburg (2003) for 2D DC resistivity. However, it was found that the interfaces in the constructed models were predominantly horizontal or vertical – the method did not want to generate dipping interfaces (Auken & Christiansen, 2004). The inclusion of diagonal spatial differences presented here attempts to remedy this problem.

Smith et al. (1999) and de Groot-Hedlin & Constable (2004) for MT, and Auken & Christiansen (2004) for DC resistivity, have developed 2D inversion procedures which generate blocky, pseudo-layered models. In their approaches, the depths of the horizontal cell boundaries are parameters in the inversion, as well as the conductivities in the cells. Constraints are placed on the lateral variability of the conductivities and the cell boundary depths. This is different from the approach taken here, which is to extend the classic minimum-structure, under-determined inversion algorithm.

Theory

General minimum-structure inversion strategy

Typical minimum-structure, under-determined inversion: (i) the subsurface is discretized using a mesh with many cells, (ii) mesh kept fixed during inversion, (*iii*) values of the physical property in the cells are the model parameters to be found via the inversion, (*iv*) seek the model which minimizes a combination of data misfit and model complexity.

The composite function to be minimized:

 $\rho(x) = (x^2 + \varepsilon^2)^{p/2}.$

Because of the squaring, minimizing the l_2 norm of a vector suppresses elements which are much larger than their neighbours. Variation of the sizes of the elements gets smeared out as much as possible.

Minimizing an l_1 -type measure of a vector does not penalize against isolated large elements.

Hence, minimizing an l_1 measure of spatial differences of a model allows for large, localized differences. This produces models with sharp interfaces between relatively uniform zones.

(See appendix below for the iterative solution procedure.)

Measure of model structure

In typical 2D minimum-structure inversions, the measure of model structure comprises a term involving finite differences in the *x*-direction and a term involving finite differences in the z-direction (and a diagonal weighting matrix). When l_2 measures are used, these two terms constitute the norm of the gradient of the model. When l_1 measures are used, the terms work independently to generate vertical and horizontal interfaces. To generate dipping interfaces, it is suggested here that terms involving finite differences in the diagonal directions be included (see Figure 1).



log₁₀ conductivity (S/m)

Figure 2: The conductivity model from which the synthetic data-set was generated. The triangles indicate the observation locations.

Three inversion results are shown below. In each inversion, the trade-off parameter was started at a relatively large value (100), and then halved at each iteration until a value was reached which resulted in a misfit sufficiently close to the target value. The trade-off parameter was then kept constant until the model and the objective function were no longer changing.

Figure 3 shows the model produced using the usual l_2 measures, and the usual x and z finite-difference terms. Figure 4 shows the model produced using the Ekblom measure with p=1, and only x and z difference terms. Figure 5 shows the model produced using the Ekblom measure (p=1) and with all four directions of finite differences.



Figure 3: The model produced using l_2 measure of model structure.



$$\Phi \,=\, \phi_d + \beta \, \phi_m.$$

Measure of data misfit:

$$\phi_d = \phi_d(\mathbf{u}), \quad \mathbf{u} = \mathbf{W}_d(\mathbf{d}^{\text{obs}} - \mathbf{d}^{\text{prd}})$$

where \mathbf{d}^{obs} is the vector of observations, \mathbf{d}^{prd} is the vector of data computed for the model, and \mathbf{W}_d is a diagonal matrix of the reciprocals of the measurement uncertainties.

Measure of structure in the model:

$$\phi_m = \sum_k \alpha_k \phi_k(\mathbf{v}_k), \quad \mathbf{v}_k = \mathbf{W}_k(\mathbf{m} - \mathbf{m}_k^{\text{ref}}),$$

where **m** is the vector of model parameters, \mathbf{W}_k are finite-difference matrices, and $\mathbf{m}_{k}^{\text{ref}}$ are possible reference models.

Figure 1: A sub-section of the 2D model mesh showing the possible directions of the finite differences.

The measure of model structure ϕ_m therefore comprises five terms, with W_1 a diagonal weighting matrix, W_2 and W_3 the usual x and z finite-difference matrices, and W_4 and W_5 finite-difference operators for diagonally up to the right and diagonally down to the right.

Figure 4: The model generated using an l_1 -type measure of model structure with the usual x and z finite-difference terms (i.e., no explicit diagonal differences).



Figure 5: The model generated using an l_1 -type measure of model structure and diagonal finite differences.

Appendix: Iterative solution procedure

The inverse problem is non-linear, both because of the non-linear relationship between data and model, and because of the non- l_2 measures. So use iterative procedure, with linearized approximation of the relationship between data and model. At the *n*th iteration, minimize

gives

 $\mathbf{R} = \operatorname{diag}\{\rho'(x_1)/x_1, \ldots, \rho'(x_N)/x_N\}$

$$\partial \phi(\mathbf{x}) = \mathbf{P}^T \mathbf{P} \mathbf{x}$$

References

Auken, E., and A.V. Christiansen, 2004, Layered and laterally constrained 2d inversion of resistivity data: Geophysics, 69, 752-761.

de Groot-Hedlin, C., and S. Constable, 2004, Inversion of magnetotelluric data for 2d structure with sharp resistivity contrasts: Geophysics, 69, 78-86.

 $\Phi^n = \phi^n_d + \beta^n \phi^n_m,$

with

and

 $\phi_d^n = \phi_d(\mathbf{u}), \qquad \mathbf{u} = \mathbf{W}_d(\mathbf{d}^{\text{obs}} - \mathbf{d}^{n-1} - \mathbf{J}\,\delta\mathbf{m}),$ $\mathbf{d}^n \approx \mathbf{d}^{n-1} + \mathbf{J}\,\delta\mathbf{m}, \qquad J_{ij} = \frac{\partial d_i}{\partial m_i},$

$$\phi_m^n = \sum_k \alpha_k \phi_k(\mathbf{v}_k), \qquad \mathbf{v}_k = \mathbf{W}_k(\mathbf{m}^{n-1} + \delta \mathbf{m} - \mathbf{m}_k^{\text{ref}}),$$

where \mathbf{m}^{n-1} is the model from the previous iteration, \mathbf{d}^{n-1} is the data for that model, $\delta \mathbf{m} = \mathbf{m}^n - \mathbf{m}^{n-1}$, and **J** is the Jacobian matrix of sensitivities computed for the model from the previous iteration.

Differentiate Φ^n with respect to δm_k , and equate resulting derivatives to zero. Differentiating the general form of the measures gives:

 $\frac{\partial \phi(\mathbf{x})}{\partial \delta m_k} = \sum_{j=1}^N \rho'(x_j) \frac{\partial x_j}{\partial \delta m_k}, \quad i.e., \qquad \frac{\partial \phi(\mathbf{x})}{\partial \delta \mathbf{m}} = \mathbf{B}^T \mathbf{q},$ where $\partial \phi / \partial \delta \mathbf{m} = \left(\partial \phi / \partial \delta m_1, \dots, \partial \phi / \partial \delta m_N \right)^T$, $B_{ij} = \partial x_i / \partial \delta m_j$, and $\mathbf{q} = (\boldsymbol{\rho}'(x_1), \dots, \boldsymbol{\rho}'(x_N))^T$. Introducing the diagonal matrix

 $= \mathbf{B}^{\star} \mathbf{K} \mathbf{X}.$ $\partial \delta \mathbf{m}$

For the measure of misfit, **B** is $-\mathbf{W}_d \mathbf{J}$. For the measure of model structure, **B** is W_k . For Ekblom's measure, the elements of **R** are:

$$R_{ii} = p \left(x_i^2 + \varepsilon^2\right)^{p/2-1}.$$

The linear system of equations to be solved at each iteration is therefore:

$$\begin{bmatrix} \mathbf{J}^T \mathbf{W}_d^T \mathbf{R}_d \mathbf{W}_d \mathbf{J} + \beta^n \sum_k \alpha_k \mathbf{W}_k^T \mathbf{R}_k \mathbf{W}_k \end{bmatrix} \delta \mathbf{m} \\ = \mathbf{J}^T \mathbf{W}_d^T \mathbf{R}_d \mathbf{W}_d (\mathbf{d}^{\text{obs}} - \mathbf{d}^{n-1}) + \beta^n \sum_k \mathbf{W}_k^T \mathbf{R}_k \mathbf{W}_k (\mathbf{m}_i^{\text{ref}} - \mathbf{m}^{n-1}).$$

The matrices **R** and the Jacobian matrix are updated at each iteration. This is know as iteratively reweighted least squares (IRLS).

For 2D MT example presented here, forward modelling done using finite difference procedure (Farquharson et al., 2002), operation of Jacobian matrix, or its transpose, on a vector done using the forward solver (e.g., Mackie and Madden, 1993, Rodi and Mackie, 2001), linear system of equations solved using direct LU decomposition or conjugate gradients.

Farquharson, C.G., and D.W. Oldenburg, 1998, Nonlinear inversion using general measures of data misfit and model structure: Geophysical Journal International, 134, 213-227.

Farquharson, C.G., and D.W. Oldenburg, 2003, Constructing piece-wise-constant models using general measures in non-linear, minimum-structure inversion algorithms: 6th International Symposium of the Society of Exploration Geophysicists of Japan, Expanded Abstracts, 240-243.

Farquharson, C.G., D.W. Oldenburg, E. Haber, and R. Shekhtman, 2002, An algorithm for the three-dimensional inversion of magnetotelluric data: 72nd Annual Meeting of the Society of Exploration Geophysicists, Expanded Abstracts, 649-652.

Loke, M.H., I. Ackworth, and T. Dahlin, 2003, A comparison of smooth and blocky inversion methods in 2d electrical imaging surveys: Exploration Geophysics, 34, 182-187.

Mackie, R.L., and T.R. Madden, 1993, Three-dimensional magnetotelluric inversion using conjugate gradients: Geophysical Journal International, 115, 215-229.

Portniaguine, O., and M.S. Zhdanov, 1999, Focusing geophysical inversion images: Geophysics, 64, 874-887.

Rodi, W., and R.L. Mackie, 2001. Nonlinear conjugate gradients algorithm for 2-d magnetotelluric inversion: Geophysics, 66, 174-187.

Smith, T., M. Hoversten, E. Gasperikova, and F. Morrison, 1999, Sharp boundary inversion of 2d magnetotelluric data: Geophysical Prospecting, 47, 469-486.