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# Inversion of the geomagnetic induction problem

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An algorithm has been found for inverting the problem of geomagnetic induction in a concentrically stratified Earth. It determines the (radial) conductivity distribution from the frequency spectrum of the ratio of internal to external magnetic potentials of any surface harmonic mode. The derivation combines the magnetic induction equation with the principle of causality in the form of an integral constraint on the frequency spectrum. This algorithm generates a single solution for the conductivity. This solution is here proved unique if the conductivity is a bounded, real analytic function with no zeros. Suggestions are made regarding the numerical application of the algorithm to real data.

## 1. INTRODUCTION

A number of investigators (Chapman 1919; Lahiri & Price 1939; Rikitake 1950; Eckhardt, Larner & Madden 1963) have made estimates of the electrical conductivity of the Earth as a function of radius by studying natural geomagnetic variations. Varying magnetic fields of external origin (i.e. those generated by electric currents in the ionosphere) induce eddy currents in the conducting Earth, which in turn produce magnetic fields of internal origin. The magnetic fields observed at the Earth's surface can be mathematically separated (Chapman & Bartels 1940) into those of external origin and those of internal origin. More precisely, one can calculate from the available experimental observations the ratio of internal and external magnetic potentials over a range of frequencies for a number of different spatial distributions of source field.

This observed induction response ratio has been used in an indirect way in the work mentioned above; the calculated behaviour of different conductivity models has been compared with the behaviour of the real Earth, and the best fitting model selected as the solution. This method provides geophysically useful information, but it lacks the elegance of a direct method. More important, one does not know if an appreciably different but untried conductivity model also fits the experimental data well.

Slichter (1933) has developed a direct method for geomagnetic deep sounding. His method deals with a horizontally stratified plane Earth, but can presumably be extended to a spherically symmetric Earth. The information required is the magnetic induction response to an infinite set of source fields with different spatial distribution, at a single frequency. This is not a very practical method because the range of source field geometries available from geomagnetic soundings is not very large. The data needed by Slichter's method are just not available.

This paper presents a direct method for determining the radial conductivity distribution. It uses the induction ratio of the earth for a single spatial distribution of source field over an infinite range of frequencies. Its central feature is the use of the principle of causality. Eckhardt (1963) and others have pointed out that, in a spherically symmetrical Earth, the magnetic field at any radius inside the Earth can be separated into internal and external parts. The internal part is always caused by the eddy currents generated by the external part; changes in the external magnetic potential must always precede the response in the internal magnetic field. This causal relationship is included in the mathematics. The result is that only one conductivity distribution can be chosen to reproduce the surface induction response and make the induction response causal at all radii. In fact, it is actually the product of conductivity and permeability which is evaluated. However, there are excellent physical grounds (Tozer 1959) for assuming unit permeability almost everywhere in the Earth. With this assumption, the conductivity itself can be evaluated.

## 2. Theoretical basis

It is first necessary to review some of the previous work on electromagnetic induction in conducting spheres. These results are taken mainly from papers by Lahiri & Price (1939) and Eckhardt (1963).

In the free space outside the sphere, any magnetic field can be represented as $-\nabla W$ where $W$ is a magnetic scalar potential satisfying Laplace's equation. In spherical polar coordinates, any solution for $W$ can be expanded as

$$W(r,\theta,\phi,t) = \sum_{n=1}^{\infty}\sum_{m=-n}^{+n} b\left[\mathscr{E}_n^m(t)\left(\frac{r}{b}\right)^n + \mathscr{I}_n^m(t)\left(\frac{r}{b}\right)^{-n-1}\right] S_n^m(\theta,\phi).$$

Here $S_n^m(\theta,\phi) = P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}$ is a surface harmonic, $b$ is the radius of the sphere and $t$ is time.

The part of the magnetic potential with coefficient $\mathscr{E}_n^m(t)$ is produced by currents outside the sphere and the part with coefficient $\mathscr{I}_n^m(t)$ is produced by currents inside the sphere. The magnetic field derived from this at the surface of the sphere is

$$\left.\begin{aligned} H_r &= \sum_{n,m} -[n\mathscr{E}_n^m(t) - (n+1)\mathscr{I}_n^m(t)]\,S_n^m(\theta,\phi),\\ H_\theta &= \sum_{n,m} -[\mathscr{E}_n^m(t) + \mathscr{I}_n^m(t)]\,\frac{\partial}{\partial\theta} S_n^m(\theta,\phi),\\ H_\phi &= \sum_{n,m} -[\mathscr{E}_n^m(t) + \mathscr{I}_n^m(t)]\,\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} S_n^m(\theta,\phi).\end{aligned}\right\} \tag{1}$$

Inside the conducting earth, $\boldsymbol{H}$ can be expressed as $\nabla\times\boldsymbol{A}$ where $\boldsymbol{A}$ is the magnetic vector potential. Lahiri & Price (1939) have shown that where the conductivity is a function of radius only, $\boldsymbol{A}$ must be of the form

$$\boldsymbol{A} = \boldsymbol{r}\times\nabla U, \tag{2}$$

where

$$\nabla^2 U = 4\pi\mu\sigma\,\partial U/\partial t.$$

These fields are poloidal. There also exist toroidal solutions, but these cannot be excited by magnetic fields of external origin and are therefore not involved in the induction problem. The permeability will henceforth be assumed unity, as noted in the introduction.

Any solution for $U$ can be written as

$$U(r,\theta,\phi,t) = \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} aF_n^m(\rho,t)\, S_n^m(\theta,\phi),$$

where

$$\frac{\partial}{\partial\rho}\left(\rho^2 \frac{\partial F_n^m}{\partial\rho}\right) = n(n+1)\, F_n^m + 4\pi a^2\rho^2\sigma(\rho)\frac{\partial F_n^m}{\partial t}, \tag{3}$$

and $a$ is the radius of the Earth, and $\rho = r/a$. The magnetic field inside the Earth is therefore

$$\left.\begin{aligned} H_r &= \sum_{n,m} -\frac{1}{\rho}\, n(n+1)\, F_n^m(\rho,t)\, S_n^m(\theta,\phi), \\ H_\theta &= \sum_{n,m} -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_n^m)\frac{\partial S_n^m}{\partial\theta}, \\ H_\phi &= \sum_{n,m} -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_n^m)\frac{1}{\sin\theta}\frac{\partial S_n^m}{\partial\phi}. \end{aligned}\right\} \tag{4}$$

Gauss (1839) showed that the magnetic field on any closed surface can be meaningfully separated into parts of external and internal origin if no current flows across the surface. Now, in an isotropic conductor, the current, the electric field (from Ohm's law) and the time derivative of the magnetic vector potential (from Maxwell's equations) lie in the same direction. Inspection of equation (2) above shows that the magnetic vector potential never has a radial component. Thus, no radial currents ever flow in a spherically symmetric Earth excited by external fields, and the Gaussian separation is valid for any (concentric) subsphere of the Earth. Equations (1) allow us to determine the internal and external coefficients $\mathscr{I}_n^m$ and $\mathscr{E}_n^m$ of the field over the surface of such a subsphere, if we are given $H_r(r,\theta,\phi,t)$ and one of $H_\theta(r,\theta,\phi,t)$ and $H_\phi(r,\theta,\phi,t)$. Let us therefore apply equations (1) to the magnetic field given by (4) and separate it into internal and external parts at an arbitrary radius $\rho a$. The result is

$$\left.\begin{aligned} \frac{n(n+1)}{\rho} F_n^m &= n\mathscr{E}_n^m - (n+1)\mathscr{I}_n^m, \\ \frac{1}{\rho}\frac{\partial}{\partial p}(\rho F_n^m) &= \mathscr{E}_n^m + \mathscr{I}_n^m. \end{aligned}\right\} \tag{5}$$

The coefficients $\mathscr{E}_n^m$ and $\mathscr{I}_n^m$ are now of course functions of $\rho$.

Equations (5) show that an external field in a given surface harmonic excites only an internal field of the same surface harmonic distribution. That is, for a given conductivity distribution, $\mathscr{I}_n^m(t)$ is a function only of $\mathscr{E}_n^m(t)$. Because of the linearity of Maxwell's equations, $\mathscr{I}_n^m(t)$ is linearly related to $\mathscr{E}_n^m(t)$. The most general linear relation is

$$\mathscr{I}_n^m(t) = \int_{-\infty}^{+\infty} K_n^m(\tau)\, \mathscr{E}_n^m(t-\tau)\,\mathrm{d}\tau, \tag{6}$$

where $K_n^m(\tau) = 0$ for $\tau < 0$ to satisfy the principle of causality. $K_n^m(\tau)$ is the impulse response of the Earth in a given surface harmonic mode.

The Fourier transform convention that will be used here is

$$g(\omega) = \int_{-\infty}^{+\infty} G(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t.$$

If $i_n^m(\omega)$, $e_n^m(\omega)$ and $k_n^m(\omega)$ denote the Fourier transforms of $\mathscr{I}_n^m(t)$, $\mathscr{E}_n^m(t)$, and $K_n^m(\tau)$ respectively, equation (6) can be written as

$$i_n^m(\omega) = k_n^m(\omega)\,e_n^m(\omega).$$

If the separation equations (5) are Fourier transformed in time, equivalent equations are obtained in $f_n^m(\rho, \omega)$, $i_n^m(\rho, \omega)$ and $e_n^m(\rho, \omega)$, where $f_n^m$ is the Fourier transform of $F_n^m$. The variables $i_n^m$ and $e_n^m$ can then be eliminated by introducing the variable $k_n^m$. The result is

$$\frac{\partial}{\partial\rho}\ln f_n^m(\rho,\omega) = -\frac{1}{\rho}\left[1+\frac{n(k_n^m(\rho,\omega)+1)}{k_n^m(\rho,\omega)-n/(n+1)}\right]. \tag{7}$$

Now equation (3) can be Fourier transformed and transposed to give:

$$2\rho\frac{\partial}{\partial\rho}\ln f_n^m(\rho,\omega)+\rho^2\left[\left(\frac{\partial}{\partial\rho}\ln f_n^m(\rho,\omega)\right)^2+\frac{\partial^2}{\partial\rho^2}\ln f_n^m(\rho,\omega)\right] = n(n+1)-4\pi a^2\rho^2\sigma(\rho)\,\mathrm{i}\omega. \tag{8}$$

Equations (7) and (8) can be combined to give

$$\frac{\partial k_n^m}{\partial\rho} = -\frac{4\pi\sigma(\rho)\,\mathrm{i}\omega a^2\rho(n+1)}{n(2n+1)}\left[k_n^m-\frac{n}{n+1}\right]^2-\frac{(2n+1)}{\rho}k_n^m. \tag{9}$$

Eckhardt derived this equation and used it to compute $k_n^m(\omega)$ numerically at the surface of the earth for different conductivity distributions (Eckhardt 1963; Eckhardt *et al.* 1963). (The different sign of $\omega$ in his derivation results from the use of a different fourier transform convention.)

This equation can be simplified with the following substitutions:

$$R_n = \frac{n}{n+1}\rho^{2n+1}, \tag{10a}$$

$$S_n = 4\left[\frac{a(n+1)}{n(2n+1)}\right]^2\rho^{-4n}\sigma, \tag{10b}$$

$$\phi_n^m = \frac{n+1}{n}R_n\left[k_n^m-\frac{n}{n+1}\right], \tag{10c}$$

when, for convenience, we drop the surface harmonic subscripts and superscripts equation (9) reduces to

$$1+\frac{\partial}{\partial R}\phi_n^m(R,\omega) = -\mathrm{i}\pi\omega S(R)\,[\phi_n^m(R,\omega)]^2. \tag{11}$$

We can now turn our attention to the constraints imposed by the causal nature of the induction process.

### 3. The role of causality

As previously noted, the impulse response function $K_n^m(\tau)$ in equation (6) must be causal in nature. That is

$$K_n^m(\tau) = 0 \quad (\tau < 0),$$

for no effect can precede its cause.

The properties of causal response functions are discussed in detail by Landau & Lifshitz (1958) and only the relevant results will be given here. The most important result is that the Fourier transform $k_n^m(\omega)$ of $K_n^m(\tau)$ must be an analytic and bounded function of $\omega$ everywhere in the upper half of the complex $\omega$ plane. The modified response function $\phi_n^m(\omega)$ must, by definition (10*c*), have the same property. It can also be shown that the real parts of $k_n^m$ and $\phi_n^m$ are even functions of $\omega$, and that the imaginary parts are odd.

Physical arguments can be used here to find the behaviour of these functions at $\omega = 0$ and as $|\omega|$ tends to infinity. At zero frequency, $k_n^m$ must necessarily be zero, since a static magnetic field cannot induce any magnetic field in a conductor with vacuum permeability. Equation (10*c*) shows that therefore $\phi_n^m = -R$ at $\omega = 0$.

As $\omega$ tends to $\pm\infty$ on the real axis, any conducting sphere tends to behave as a superconducting sphere for which

$$k_n^m = n/(n+1).$$

This is the value of $k_n^m$ required to make $H_r = 0$ in equation (1). It must, of course, be assumed that the Earth's conductivity is not zero at the radius where $k_n^m$ is evaluated.

Therefore,
$$\lim_{\omega\to\pm\infty} \phi_n^m = 0.$$

Furthermore, $\phi_n^m$ must tend to this same limit as $|\omega|$ tends to infinity anywhere in the upper half plane because it is analytic and bounded everywhere there. Exactly how fast $\phi_n^m$ tends to zero as $|\omega|$ tends to infinity can also be deduced from physical arguments. As the frequency increases, the skin depth of magnetic field penetration must eventually become much less than the radius difference in which the conductivity changes appreciably. At frequencies greater than this, $\phi_n^m$ must tend to that of a uniform sphere whose conductivity equals the surface conductivity of the actual sphere. The solution for a uniform sphere (Chapman & Price 1930) shows that $\phi_n^m$ tends to zero as $(-\mathrm{i}\pi\omega S)^{-\frac{1}{2}}$, where $S$ is the 'modified conductivity' defined in equation (10*b*).

The properties of $\partial\phi_n^m/\partial R$ follow from these results and equation (11). The function $\partial\phi_n^m/\partial R$ is analytic and bounded in the upper half $\omega$ plane and tends to zero as $|\omega|$ tends to infinity in the upper half plane.

The analyticity and boundedness of $\partial\phi_n^m/\partial R$ in the upper half $\omega$ plane permits the use of Cauchy's integral formula. That is,

$$\frac{\partial\phi_n^m(R,\omega)}{\partial R} = \frac{1}{2\pi\mathrm{i}}\int_C \frac{1}{\omega'-\omega}\,\frac{\partial\phi_n^m(R,\omega')}{\partial R}\,\mathrm{d}\omega',$$

where no part of the contour $C$ lies in the lower half plane and $\omega$ is inside the contour $C$. If $\omega$ is on the contour $C$, Cauchy's integral formula becomes

$$\frac{\partial\phi_n^m(R,\omega)}{\partial R} = \frac{1}{\pi \mathrm{i}} ⨍_C \frac{1}{\omega'-\omega}\frac{\partial\phi_n^m(R,\omega')}{\partial R}\,\mathrm{d}\omega'. \tag{12}$$

The bar across the integral sign indicates that the Cauchy principle value of the integral is to be taken.

Equation (12) will be applied to the case where $\omega = 0$ and the contour $C$ is as shown in figure 1. The radius of the semicircle is allowed to tend to infinity.

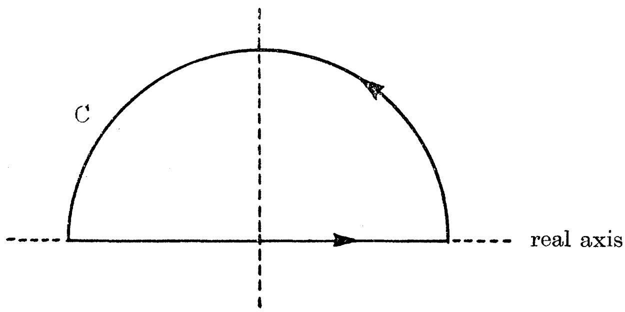


FIGURE 1

Since $\partial\phi_n^m/\partial R$ tends to zero as $|\omega|$ tends to infinity ($\operatorname{Im}\omega \geqslant 0$), then $\partial\phi_n^m/\omega\,\partial R$ tends to zero faster than $\omega^{-1}$. The integral over the semi-circular part of the contour must vanish. Equation (12) reduces to

$$\left.\frac{\partial\phi_n^m(R,\omega)}{\partial R}\right|_{\omega=0} = \frac{1}{\pi} ⨍_{-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega}\frac{\partial\phi_n^m(R,\omega)}{\partial R}\,\mathrm{d}\omega.$$

However, since by equation (11) $\partial\phi_n^m/\partial R|_{\omega=0} = -1$, this can be written as an integral constraint on $\partial\phi_n^m/\partial R$, i.e.

$$-\pi = ⨍_{-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega}\frac{\partial\phi_n^m(R,\omega)}{\partial R}\,\mathrm{d}\omega. \tag{13}$$

Equation (13) must be satisfied by any physically realistic response function at any depth. They can be applied to the modified Eckhardt equation (11) to see what condition the conductivity must satisfy for $\phi_n^m$ to remain causal at all radii.

## 4. The conductivity integral

Equation (11) can be written as

$$\frac{\partial}{\partial R}\left(\frac{\phi_n^m}{\mathrm{i}\omega}\right) + \frac{1}{\mathrm{i}\omega} = -\pi S(\phi_n^m)^2.$$

Integration of this equation with respect to $\omega$ over the range $-\infty$ to $+\infty$ yields

$$\int_{-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega} \frac{\partial \phi_n^m}{\partial R} \,\mathrm{d}\omega + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{\mathrm{i}\omega} = -\pi S \int_{-\infty}^{+\infty} (\phi_n^m)^2 \,\mathrm{d}\omega. \tag{14}$$

The second term on the left-hand side is zero.

When we apply (13), equation (14) becomes

$$1 = S \int_{-\infty}^{+\infty} (\phi_n^m)^2 \,\mathrm{d}\omega. \tag{15}$$

This becomes

$$S(R) = \left[ 2 \int_0^{\infty} \mathrm{Re}\,(\phi_n^m(R,\omega))^2 \,\mathrm{d}\omega \right]^{-1}, \tag{16}$$

by utilizing the symmetry and boundedness of $(\phi_n^m)^2$. Since $S$ is, in effect, the conductivity (from (10*b*)), the local conductivity has been evaluated in terms of the magnetic response function $\phi_n^m$. Specifically, the conductivity is

$$\sigma = \frac{1}{4} \left[ \frac{(2n+1)\,n}{a(n+1)} \right]^2 \rho^{4n} \left[ 2 \int_0^{\infty} \mathrm{Re}\,(\phi_n^m)^2 \,\mathrm{d}\omega \right]^{-1}. \tag{17}$$

Strictly speaking, the $\sigma$ given by (17) is not $\sigma(\rho)$; it is the conductivity immediately below $\rho$. Mathematically speaking, it is $\lim\,(\epsilon \to 0)\,\sigma(\rho - \epsilon)$. Thus if $\phi_n^m$ is known at a discontinuity in $\sigma$ (e.g. at the surface of the Earth), the formula (17) evaluates the conductivity on the lower side of the discontinuity. A divergent integral in (17) corresponds to zero conductivity.

## 5. The final equation for $\phi_n^m$

Equation (16) can be used to eliminate the modified conductivity $S$ from equation (11), giving

$$\frac{\partial \phi_n^m}{\partial R} + 1 = -\mathrm{i}\pi\omega(\phi_n^m)^2 \left[ 2 \int_0^{\infty} \mathrm{Re}\,(\phi_n^m)^2 \,\mathrm{d}\omega \right]^{-1}. \tag{18}$$

This is a single nonlinear partial integrodifferential equation (first order in the independent variable $R$) in one unknown, $\phi_n^m\,(R,\omega)$. This is the equation from which a complete solution for the problem can be obtained from the boundary condition that $\phi_n^m(R,\omega)$ is specified for all $\omega$ at some $R$ (i.e. the surface of the Earth). Unfortunately there are no general mathematical methods for dealing with this type of equation. However, it does lend itself to numerical iteration in $R$. Once a solution for $\phi_n^m(R,\omega)$ has been obtained numerically, the conductivity function can be obobtained from it by using equation (17).

It must still be proved, however, that knowledge of $\phi_n^m(R,\omega)$ for all $\omega$ at some $R$ is enough to determine a unique solution $\phi_n^m(R,\omega)$ for all $\omega$, and all $R$,

$$0 < R \leqslant n/(n+1).$$

This is done in the next section.

## 6. Uniqueness

A proof of the uniqueness of solutions generated from completely arbitrary surface functions of $\phi_n^m(R,\omega)|_{R_0}$ has not been obtained. The uniqueness theorem given here restricts itself to the set of all $\phi_n^m(R,\omega)|_{R_0}$ generated by conductivities for which the conductivity integral

$$I = S^{-1} = \int_{-\infty}^{+\infty} (\phi_n^m)^2 \,\mathrm{d}\omega,$$

is a non zero real analytic function. That is, we must be able to write

$$I = \sum_{n=0}^{\infty} a_n (R-R_0)^n \quad \left(0 < R \leqslant R_0 = \frac{n}{n+1}\right),$$

where $I$ has no zeros in the interval $0 < R \leqslant R_0$. In effect, only non-zero bounded conductivity functions that are infinitely differentiable are considered.

With this sort of conductivity function, equation (11) must generate a $\phi_n^m$ that is continuous in $R$ and all of whose derivatives are continuous in $R$. Thus $[\phi_n^m(R,\omega)]^2$ can be expanded in a Taylor series in $R$ about $R_0$ and integrated over $\omega$. That is

$$I(R) = \int_{-\infty}^{+\infty} (\phi_n^m)^2 |_{R_0} \mathrm{d}\omega + (R-R_0) \int_{-\infty}^{+\infty} \left.\frac{\partial(\phi_n^m)^2}{\partial R}\right|_{R_0} \mathrm{d}\omega + \tfrac{1}{2}(R-R_0)^2 \int_{-\infty}^{+\infty} \left.\frac{\partial^2(\phi_n^m)^2}{\partial R^2}\right|_{R_0} \mathrm{d}\omega + \ldots.$$

Now each of the derivatives of $(\phi_n^m)^2$ with respect to $R$ can be evaluated in terms of $\phi_n^m(\omega, R)|_{R_0}$ by means of repeated applications of the recursion relation

$$\frac{\partial \phi_n^m}{\partial R} = -1 - \frac{\mathrm{i}\pi\omega(\phi_n^m)^2}{I},$$

derived from the differential equation (11). For example, the first few terms in the Taylor series for $(\phi_n^m)^2$ are evaluated as follows:

$$(\phi_n^m)^2 = (\phi_n^m)^2,$$

$$\frac{\partial(\phi_n^m)^2}{\partial R} = -2\phi_n^m \left[1 + \frac{\mathrm{i}\pi\omega(\phi_n^m)^2}{I}\right],$$

$$\frac{\partial^2(\phi_n^m)^2}{\partial R^2} = 2\phi_n^m \left[\frac{2\mathrm{i}\pi\omega\phi_n^m}{I}\left(1 + \frac{\mathrm{i}\pi\omega(\phi_n^m)^2}{I}\right) - \frac{\mathrm{i}\pi\omega(\phi_n^m)^2}{I^2}\frac{\partial I}{\partial R}\right] + 2\left[1 + \frac{\mathrm{i}\pi\omega(\phi_n^m)^2}{I}\right]^2.$$

It can be seen that the only terms appearing in the formulae for the derivatives of $(\phi_n^m)^2$ are $\phi_n^m$ and a set $I$, $\partial I/\partial R$, $\partial^2 I/\partial R^2$, etc. Any such term $\partial^k I/\partial R^k$ first occurs in the formula for $\partial^{k+1}(\phi_n^m)^2/\partial R^{k+1}$, which means that $\partial^k(\phi_n^m)^2/\partial R^k$ has already been evaluated and that $\partial^k I/\partial R^k$ is already known as

$$\int_{-\infty}^{+\infty} \frac{\partial^k(\phi_n^m)^2}{\partial R^k}.$$

Thus all the terms of the power series expansion for $I$ can be calculated from $\phi_n^m(\omega, R)|_{R_0}$, and because two different functions cannot have the same power series, it follows that the solution for $I$ and thus for the conductivity, is unique.

A generalization of this proof to less restricted conductivity functions may be possible. One restriction that cannot be removed, however, is that the conductivity has a bounded integral. It is a physical fact that any distribution of conductivity inside a shell of infinite integrated conductivity may be taken as a solution without conflicting with the surface boundary condition of specified $\phi_n^m(\omega)$. Essentially the same uniqueness theorem may also be proved using the theory of inverse Sturm-Liouville problems as developed by Borg (1945) and others.

It is interesting to note that knowledge of $\phi_n^m(\omega)$ at the surface of the earth in a single surface harmonic mode is enough to give a unique solution for the conductivity.

## 7. Conclusions

There is no reason for believing that this algorithm can, from real data, evaluate the Earth's conductivity more accurately than trial and error methods have already done. Its real significance to geophysics is that it generates unique solutions (subject to certain restrictions). It shows for the first time that surface geomagnetic measurements are, in fact, sufficient in principle to determine the real conductivity distribution.

The basic method is equally applicable to a number of related problems. The corresponding problem for a plane stratified earth, for example, may be solved. In this case the first order equation equivalent to equation (9) is (using analogous notation):

$$\frac{\partial k_\lambda}{\partial z} = -\frac{4\pi}{\lambda} k_\lambda - \mathrm{i}\omega\lambda\sigma(z)\,[k_\lambda - 1]^2, \tag{19}$$

and the conductivity formula equivalent to equation (17) is

$$\sigma = \frac{4\pi^2}{\lambda^2}\left[2\int_0^\infty \mathrm{Re}\,(k_\lambda - 1)^2\,\mathrm{d}\omega\right]^{-1}. \tag{20}$$

Here $\lambda$ is the horizontal wavelength of the (sinusoidal) exciting field.

This sort of problem requires, however, that the surface of the conducting body define a suitable orthogonal coordinate system, and that the conductivity be a function only of the coordinate normal to the surface of the body. These conditions are necessary for the derivation of the analogues of Eckhardt's equation (9). These points are discussed in more detail by Eckhardt (1968).

Price (1962) and Watanabe (1964) have shown that the geomagnetic deep sounding and magnetotelluric problems are mathematically equivalent. If one can be inverted, the other can. Therefore the magnetotelluric method can produce unique solutions in principle. The algorithm of this paper may be applied to a magnetotelluric problem by converting it to the corresponding geomagnetic one.

It is hoped to apply this algorithm to real Earth data. The conductivity of the

Earth is probably finite, non-zero and more or less continuous. In principle, therefore, it is uniquely recoverable from surface measurements. A major obstacle is the impossible requirement that $\phi_n^m(\omega, R)|_{R_0}$ be known over an infinite frequency range. If $\phi_n^m$ is known (for some $n, m$) over a fairly wide frequency range, however, reasonably accurate extrapolation functions can be fitted to the high and low frequency limits of the data. These extrapolation functions must, of course, have the theoretically required asymptotic behaviour and symmetry properties. It is then fairly straightforward to evaluate numerically the conductivity integral in equation (18) and integrate this equation downwards into the earth from the surface, again numerically. The conductivity integral must of course be re-evaluated continually in this process.

There is another problem to solve in treating real data. The conductivity of the Earth is not radially symmetric. There are large horizontal inhomogeneities at the surface in the form of oceans and continental structures. In principle, the effects of these could be computed and removed from the data. In practice, it is much easier to settle for an approximate solution and throw away the data at frequencies above, say, a few cycles per day. The data at frequencies below this are not significantly affected by the surface inhomogeneities; the data at frequencies above this can be filled in by the extrapolation procedure described above.

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