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Can a 2-D MT frequency response always be interpreted as a 1-D response?

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SUMMARY

Weidelt and Kaikkonen showed that in the transverse magnetic (TM) mode of magnetotellurics it is not always possible to match exactly the 2-D response at a single site with a 1-D model, although a good approximation usually seems possible. We give a new elementary example of this failure. We show for the first time that the transverse electric (TE) mode responses can also be impossible to match with a 1-D response, and that the deviations can be very large.

Key words: Electromagnetic theory; Magnetotelluric; Geomagnetic induction.

1 INTRODUCTION

The properties that characterize the magnetotelluric frequency response of a perfectly layered, 1-D structure are thoroughly understood. Weidelt (1972) showed that the Schmucker complex admittance function, defined by $c(\omega) = E_x/\mathrm{i}\,\omega B_y = -E_y/\mathrm{i}\,\omega B_x$ can always be expressed in terms of non-decreasing spectral function $a(\lambda)$ by a Stieltjes integral

$$c(\omega) = a_0 + \int_0^\infty \frac{\mathrm{d}a(\lambda)}{\lambda + \mathrm{i}\,\omega}, \tag{1}$$

which leads to a host of conditions (for example, Weidelt 1986; Yee & Paulson 1988). If the conducting structure has finite vertical extent and $\sigma(z)$ has a finite integral (surely a reasonable requirements for a model of the Earth), this simplifies to

$$c(\omega) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n + \mathrm{i}\,\omega} \tag{2}$$

with positive numerators $a_n > 0$ and a set of positive λ_n which increase without bound. Eq. (2) is a special case of (1) when the function $a(\lambda)$ becomes a rising staircase with jumps of a_n at $\lambda = \lambda_n$.

What characterizes a finite set of values drawn from the function $c(\omega)$ at the frequencies ω_j, $j = 1, 2, \ldots N$, the kind of collection made available by actual observation? Parker (1980) gave necessary and sufficient conditions based on (2), in terms of the feasibility of a semi-infinite linear program, and Parker & Whaler (1981) extended the idea to uncertain observations with Gaussian statistics, by means of a quadratic program. Parker & Booker (1996) gave a treatment valid when the measurements are represented by apparent resistivity and phase values, rather than the real and imaginary parts of $c(\omega_j)$. With these results we can determine with confidence whether or not a given finite collection of (noisy or exact) MT observations is compatible with a 1-D model. The programs for performing the test are fast and completely reliable. As a preliminary step before embarking on modelling or drawing inferences about the Earth, this kind of test is invaluable.

When we turn to 2-D conductivity systems, far less is known. Suppose the observations can be converted into TM and TE mode admittances, a first step if a 2-D model is to be possible. If it were true that the frequency response (TE and TM separately) at each site could always be matched by a 1-D model under the site, this would be a very helpful quality control tool, since 1-D testing is so effective. Parker & Booker (1996) suggested this idea, while acknowledging that Weidelt & Kaikkonen (1994) had shown that with TM data such a representation is in fact not always possible. Indeed, Weidelt & Kaikkonen gave several counterexamples. They also provided a number of necessary conditions, including that, in TM mode induction, the phase of $c(\omega)$ can never fall outside the interval $(-\frac{1}{2}\pi, 0)$, consistent with the behaviour of 1-D systems. That work leaves open the question of whether TE responses can be tested against the 1-D model.

In this paper, we close the gap by showing that there are indeed TE responses at a single site incompatible with a 1-D interpretation. For this purpose we develop a new method, valid at all frequencies, to solve the TE induction problem in a thin surface layer. We also describe a very simple counterexample for the TM case. It appears that TM mode responses at a single site can usually (and perhaps always) be approximated remarkably well with functions derived from 1-D conductivities. That is manifestly not the case for the TE mode problem.

2 A SIMPLE TM RESPONSE

The analysis of Weidelt & Kaikkonen (1994) is based on the solution to the problem of TM mode (also called B-polarization) induction in which the spectral function $a(\lambda)$ and (1) are required. We provide a more elementary example.

Consider a uniform conductor, conductivity σ_0, infinite in the y direction, and confined to the rectangular region Ω given by

$$\Omega = 0 \le x \le a \quad \cap \quad 0 \le z \le b, \tag{3}$$

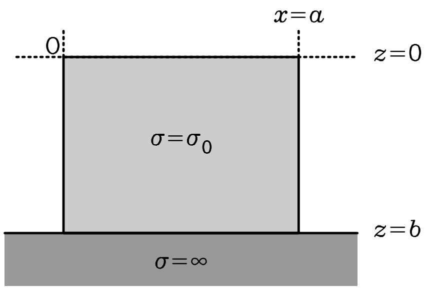

Figure 1. A vertical section through the model conducting system.

where z is positive downward, and $z = 0$ is the Earth's surface. The body is bounded on the bottom by perfect a conductor, and on the sides by perfect insulators; see Fig. 1. Electromagnetic induction is driven in this system by a uniform, horizontal magnetic field in the y direction, oscillating as $e^{\,i\omega t}$. Because the source field is parallel to the strike of the structure, this is by definition TM mode induction. Then, as is well known (see, for example, Weaver 1994) the magnetic field within the body is of the form $\mathbf{B} = \hat{\mathbf{y}} B(x, z) e^{i\omega t}$ and, since the conductivity is uniform, the complex scalar function B obeys the partial differential equation

$$\nabla^2 B = i\,\omega\mu_0\sigma_0 B. \tag{4}$$

The boundary conditions for this equation are easily verified to be

$$B = B_0, \quad \text{on} \quad z = 0 \tag{5}$$

$$B = B_0, \quad \text{on} \quad x = 0, \quad 0 \le z \le b \tag{6}$$

$$B = B_0, \quad \text{on} \quad x = a, \quad 0 \le z \le b \tag{7}$$

$$\frac{\partial B}{\partial z} = 0, \quad \text{on} \quad z = b. \tag{8}$$

The choice of domain Ω and the assumption of constant conductivity allows us to solve (4) by separation of variables in the classical way as a product of trigonometric or exponential functions. To satisfy (6) and (7) we set

$$B(x, z) = B_0 + \sum_{m=1}^{\infty} A_m(z) \sin\left(\frac{m\pi x}{a}\right). \tag{9}$$

Inserting this expression into (4) gives

$$\sum_{m=1}^{\infty} \left(A_m''(z) - k_m^2 A_m(z)\right) \sin\left(\frac{m\pi x}{a}\right) = i\,\omega\mu_0\sigma_0 B_0, \tag{10}$$

where $' = d/dz$ and

$$k_m = \left(\frac{m^2\pi^2}{a^2} + i\,\omega\mu_0\sigma_0\right)^{\frac{1}{2}}. \tag{11}$$

On each horizontal line (10) is the Fourier sine series for a constant; evaluating the coefficients in this series, we find

$$A_m''(z) - k_m^2 A_m(z) = \begin{cases} \dfrac{4i\,\omega\mu_0\sigma_0 B_0}{m\pi}, & m \text{ odd} \\ 0, & m \text{ even.} \end{cases} \tag{12}$$

These differential equations have the general solutions

$$A_m(z) = -\frac{4i\,\omega\mu_0 B_0}{m\pi k_m^2}\epsilon_m + \alpha_m \sinh k_m z + \beta_m \cosh k_m z, \tag{13}$$

where $\epsilon_m = m(\mathrm{mod}\,2)$, that is, either 1 or 0 as m is odd or even. Finally, we apply the boundary conditions on the top and bottom surfaces, (5) and (8), having first substituted (13) into (9). After straightforward algebra, we obtain

$$B(x, z) = B_0 + \sum_{\text{odd } m \ge 1} \frac{4i\,\omega\mu_0 B_0}{m\pi k_m^2} \times \left[\frac{\cosh k_m(b - z)}{\cosh k_m b} - 1\right] \sin\left(\frac{m\pi x}{a}\right) \tag{14}$$

which is the solution to the boundary value problem.

Our principal interest is in the frequency response of the TM mode at the surface, which is

$$c(\omega, x) = \frac{E_x}{i\,\omega B_y} = \frac{1}{i\omega\mu_0\sigma_0 B_0}\left(\frac{\partial B}{\partial z}\right)_{z=0} \tag{15}$$

$$= \frac{4b}{\pi} \sum_{\text{odd } m \ge 1} \frac{\tanh k_m b}{m k_m b} \sin\left(\frac{m\pi x}{a}\right). \tag{16}$$

To determine for a fixed x whether the function in (16) can be interpreted as the admittance of a 1-D conductor, we write (16) as a spectral expansion like (2). First note the Mittag–Leffler expansion (Gradshteyn & Ryzhik 1965)

$$\frac{\tanh z}{z} = \frac{8}{\pi^2} \sum_{\text{odd } n \ge 1} \frac{1}{n^2 + 4z^2/\pi^2}. \tag{17}$$

Substituting (17) into (16) gives the spectral expansion

$$c(\omega, x) = \sum_{\text{odd } m \ge 1} \sum_{\text{odd } n \ge 1} \frac{8(m\pi\mu_0\sigma_0 b)^{-1} \sin(m\pi x/a)}{i\,\omega + \frac{\pi^2}{\mu_0\sigma_0}\left(\frac{m^2}{a^2} + \frac{n^2}{4b^2}\right)} \tag{18}$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{i\,\omega + \lambda_k}. \tag{19}$$

We see immediately from (18) that some of the numerators in the spectral expansion (19) must be negative for any value of x because of the sine function. Thus the admittance can never be generated from a 1-D profile, and exact 1-D interpretation of the responses is impossible.

We illustrate the behaviour of $a(\lambda)$ by constructing the spectral function equivalent of (19) for a particular set of parameters. To find the values in (19) we sort the set of numbers $\{(m^2/a^2 + n^2/4b^2)\,\pi^2/\mu_0\sigma_0\}$ into a non-decreasing sequence, and assign λ_k to the kth member, and a_k to the associated numerator; if there are repeated values of λ_k, we sum the associated numerators to form a_k. Then we construct the spectral function as the piecewise constant function with steps of a_k at λ_k. In the numerical example we take $a = 2b$ and evaluate the response at $x = \frac{1}{2}a$. As noted in the introduction, spectral functions based on 1-D systems resemble a rising staircase; Fig. 2 shows how far (18) departs from that paradigm.

While this example vividly demonstrates that c cannot be reproduced exactly, we might wonder how well it can be approximated. Fig. 3 shows the error in a 1-D response approximation obtained by the D^+ method (Parker & Whaler 1981), based on a sampling of the admittance at 50 frequencies, spaced logarithmically. The D^+ model comprises a set of delta functions in conductivities, and gives the best (weighted) 2-norm fit to the admittance in the class of 1-D profiles. The discrepancy is tiny, undetectable by observation. Equally remarkable, in view the complicated form of the true spectral function, is the fact that $a(\lambda)$ for the 1-D model has only seven steps.

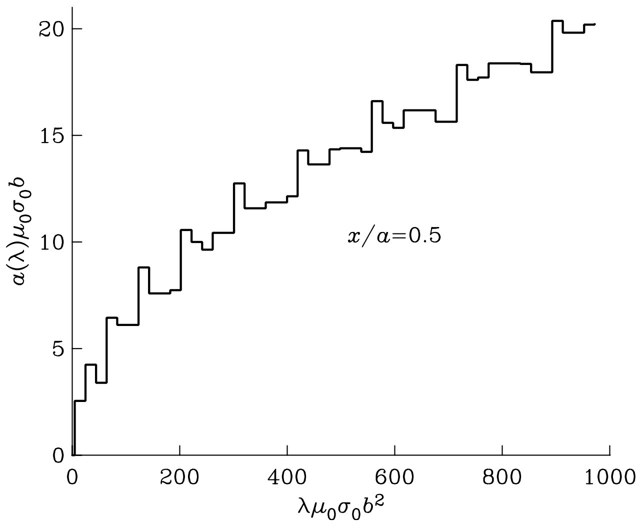

Figure 2. The spectral function derived from (18) for $a = 2b$ and evaluated at $x = \frac{1}{2}a$. The variables plotted are dimensionless.

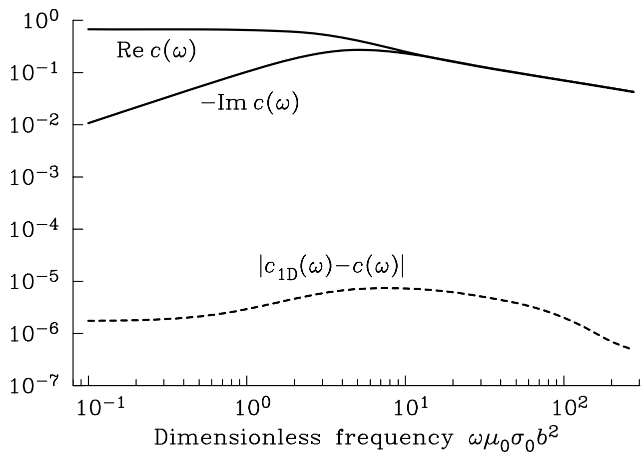

Figure 3. Admittance of the model at $x = \frac{1}{2}a$ with $a = 2b$, and error in a 1-D response fitted to the true admittance at 50 frequencies in the interval shown. All values scaled by $1/b$.

3 AN INCOMPATIBLE TE RESPONSE

The comparative simplicity of the response (16) and the subsequent analysis owes much to the presence of the perfectly conducting base in the model. Weidelt showed (1972) that a spherically symmetric conducting system can be exactly represented by an equivalent 1-D layered model terminated by a perfect conductor, which suggests that such systems are natural even in flat-Earth approximations. If we confine ourselves to systems of this kind, there is an intuitively appealing test for one-dimensionality of the admittance: every admittance value must fall within a semicircular region in the complex plane:

$$c(\omega) \in \mathcal{Z} \text{ where } \mathcal{Z} = \text{Im}\, z \le 0 \quad \cap \quad \left| z - \frac{1}{2}b \right| \le \frac{1}{2}b. \tag{20}$$

As before b is the depth to the base. If an entire 1-D response function is plotted in the complex c plane, a simple curve results that starts at $c(0) = b$, the low frequency limit and, provided the conductivity at the surface does not vanish, arrives at $c(\infty) = 0$; the curve remains inside $\mathcal{Z}$ for all ω; see Fig. 4. If an admittance function

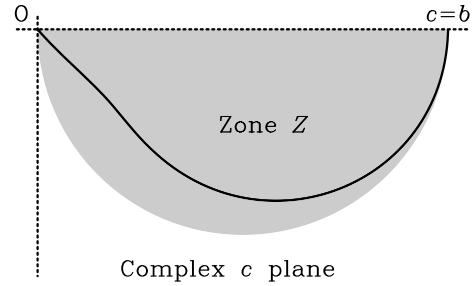

Figure 4. The zone $\mathcal{Z}$ in the complex c plane. The smooth curve is the locus of the complete admittance function for a uniformly conducting, 1-D layer.

strays outside the zone, it is incompatible with the assumptions. The test is only a necessary condition. Many TE and TM responses remain inside of course, including the TM response derived in the previous section. A proof that $c(\omega) \in \mathcal{Z}$ for 1-D structures is given in the Appendix.

We come finally to a TE mode system that violates (20) at almost every frequency. It consists of a variable conductivity, thin sheet at the surface, over an insulating layer, terminated by a perfect conductor. The driving magnetic field is in the x direction, periodic in time as before. Currents $j(x)$ flowing in the sheet in the y direction generate a magnetic field; the perfect conductor at $z = b$ does not allow vertical magnetic fields there, and can thus be simulated by image currents $-\ j(x)$ confined to the depth $z = 2b$. The conductivity is concentrated into a thin layer in such away that the conductivity-thickness product, the conductance, τ is finite. We write

$$\tau(x) = \tau_0 + \Delta\tau(x), \tag{21}$$

where τ_0 is constant. Let the electric field in the sheet be $e(x) = E_y(x, 0)$. It too is split into a constant and a variable part

$$e(x) = e_0 + \Delta e(x). \tag{22}$$

We will assume that $\|\Delta\tau\|_2$ is finite and then we can find e_0 from the 1-D theory

$$e_0 = \frac{\mathrm{i}\,\omega b B_0}{1 + \mathrm{i}\,\omega\mu_0\tau_0 b}, \tag{23}$$

where B_0 is the magnitude of the source field. From the induction Maxwell equation on $z = 0$ we have

$$\hat{\mathbf{z}} \cdot \nabla \times \mathbf{E} = -\mathrm{i}\,\omega\hat{\mathbf{z}} \cdot (\hat{\mathbf{x}}B_0 + \mathbf{B}_J) = -\mathrm{i}\,\omega\hat{\mathbf{z}} \cdot \mathbf{B}_J, \tag{24}$$

where $\mathbf{B}_J$ is the part of the magnetic field due to induced currents. Applying the Biot–Savart law to (24) we find that

$$\frac{\mathrm{d}\Delta e}{\mathrm{d}x} = -\mathrm{i}\,\omega \int_{-\infty}^{\infty} G(x - x')j(x')\,\mathrm{d}x', \tag{25}$$

where G includes the fields from the image currents

$$G(x) = -\frac{\mu_0}{2\pi}\left(\frac{1}{x} - \frac{x}{4b^2 + x^2}\right). \tag{26}$$

The integrals here are understood to be principal part integrals. By Ohm's law $j(x) = \tau(x)e(x)$; inserting this and breaking $\tau(x)$ and $e(x)$ apart with (21) and (22), we find

$$\begin{aligned}\frac{\mathrm{d}\Delta e}{\mathrm{d}x} &= -\mathrm{i}\,\omega e_0 \int_{-\infty}^{\infty} G(x - x')\Delta\tau(x')\,\mathrm{d}x' \\ &\quad - \mathrm{i}\,\omega \int_{-\infty}^{\infty} G(x - x')\tau(x')\Delta e(x')\,\mathrm{d}x' \end{aligned} \tag{27}$$

since $\int G(x)\,\mathrm{d}x = 0$. We will assume that Δe vanishes at infinity, and then we can integrate (27) over $(-\infty, x)$ to obtain a Fredholm integral equation of the second kind for the electric field perturbation

$$\Delta e(x) = g(x) + \mathrm{i}\,\omega \int_{-\infty}^{\infty} F(x - x')\tau(x')\Delta e(x')\,\mathrm{d}x' \tag{28}$$

with

$$F(x) = \frac{\mu_0}{4\pi} \ln \frac{x^2}{4b^2 + x^2} \tag{29}$$

and

$$g(x) = \mathrm{i}\,\omega e_0 \int_{-\infty}^{\infty} F(x - x')\Delta\tau(x')\,\mathrm{d}x'. \tag{30}$$

We will solve (28) iteratively in the Fourier domain because numerical Fourier transforms are so efficient, and also because the convolution operation is mapped into a simple multiplication. Define

$$\hat{f}(k) = \mathcal{F}[f] = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi \mathrm{i}kx}\,\mathrm{d}x. \tag{31}$$

Then (28) is transformed into

$$\hat{\Delta}e(k) = \hat{g}(k) + \mathrm{i}\,\omega\hat{F}(k)\mathcal{F}[\tau\Delta e], \tag{32}$$

where

$$\hat{F}(k) = -\mu_0 \frac{1 - \mathrm{e}^{-4\pi|k|b}}{4\pi|k|}; \quad \hat{g}(k) = \mathrm{i}\,\omega e_0 \hat{F}(k)\hat{\Delta}\tau(k). \tag{33}$$

A standard technique for (28) is the Neumann series (Porter & Stirling 1990), which is equivalent to a fixed point iteration; its Fourier counterpart is the scheme

$$\hat{\Delta}e_{n+1} = \hat{g} + \mathrm{i}\,\omega\hat{F}\mathcal{F}\left[\tau\mathcal{F}^{-1}[\hat{\Delta}e_n]\right], \quad n = 0, 1, 2, \ldots \tag{34}$$

with an initialization $\hat{\Delta}e_0 = 0$. However, it is well known that the convergence of the Neumann series is limited to $|\omega| < \omega_*$, a critical value, and we would like to examine solutions at any real frequency. Therefore, we modify the scheme as follows: in (32) we expand τ with (21) and rearrange

$$\hat{\Delta}e = \frac{\hat{g}}{1 - \mathrm{i}\,\omega\tau_0\hat{F}} + \frac{\mathrm{i}\,\omega\hat{F}}{1 - \mathrm{i}\,\omega\tau_0\hat{F}}\mathcal{F}[\Delta\tau\Delta e]. \tag{35}$$

A fixed-point iteration based on this equation converges for every ω provided certain conditions obtain on τ as we will now prove.

We make use of two norms, the 2-norm which we write $\|\cdot\|$ and the sup norm $\|\cdot\|_\infty$. Then it is elementary that $\|\mathcal{F}[f]\| = \|f\|$ and $\|g \cdot f\| \le \|g\|_\infty \cdot \|f\|$. Abbreviate (35) as

$$\hat{\Delta}e = \hat{g}_1 + \hat{F}_1\mathcal{F}[\Delta\tau\Delta e] \tag{36}$$

and then a step in the fixed point iterative procedure is

$$\hat{\Delta}e_{n+1} = \hat{g}_1 + \hat{F}_1\mathcal{F}\left[\Delta\tau\mathcal{F}^{-1}[\hat{\Delta}e_n]\right]. \tag{37}$$

Thus

$$\|\hat{\Delta}e_{n+1} - \hat{\Delta}e_n\| = \left\|\hat{F}_1\mathcal{F}\left[\Delta\tau\mathcal{F}^{-1}[\hat{\Delta}e_n - \hat{\Delta}e_{n-1}]\right]\right\| \tag{38}$$

$$\le \|\hat{F}_1\|_\infty \cdot \|\Delta\tau\mathcal{F}^{-1}[\hat{\Delta}e_n - \hat{\Delta}e_{n-1}]\| \tag{39}$$

$$\le \|\hat{F}_1\|_\infty \cdot \|\Delta\tau\|_\infty \cdot \|\hat{\Delta}e_n - \hat{\Delta}e_{n-1}\|. \tag{40}$$

Therefore, the process is convergent provided that

$$\|\hat{F}_1\|_\infty \cdot \|\Delta\tau\|_\infty = \left\|\frac{\mathrm{i}\,\omega\hat{F}}{1 - \mathrm{i}\,\omega\tau_0\hat{F}}\right\|_\infty \cdot \|\Delta\tau\|_\infty < 1. \tag{41}$$

A short calculation shows that

$$\left\|\frac{\mathrm{i}\,\omega\hat{F}}{1 - \mathrm{i}\,\omega\tau_0\hat{F}}\right\|_\infty = \frac{\omega\mu_0 b}{(1 + \omega^2\mu_0^2 b^2\tau_0^2)^{\frac{1}{2}}} \tag{42}$$

from which it follows that convergence of (37) is assured at all real frequencies when $\|\Delta\tau\|_\infty/\tau_0 < 1$. [Q.E.D.]

To evaluate the admittance we need the x component of the magnetic field which, unlike the surface field in TM induction, varies with x. Above the sheet the electric field E_y is harmonic, and hence the spatially variable part can be found from the surface electric field by upward continuation

$$\Delta E_y(x, z) = \int_{-\infty}^{\infty} \mathrm{e}^{+2\pi|k|z}\mathrm{e}^{2\pi \mathrm{i}kx}\,\hat{\Delta}e(k)\,\mathrm{d}k, \tag{43}$$

where the perhaps surprising sign of z in the exponent is the result of the convention that z increases downward, and $z < 0$ above the sheet. Hence

$$\left(\frac{\partial\Delta E_y}{\partial z}\right)_{z\uparrow 0} = \int_{-\infty}^{\infty} 2\pi|k|\mathrm{e}^{2\pi \mathrm{i}kx}\,\hat{\Delta}e(k)\,\mathrm{d}k = \mathcal{F}^{-1}[2\pi|k|\hat{\Delta}e]. \tag{44}$$

The x component of $\nabla \times \mathbf{E} = -\mathrm{i}\,\omega\mathbf{B}$ gives us $B_x = (\mathrm{i}\,\omega)^{-1}\partial E_y/\partial z$ and together with (44), this provides us with the convenient expression

$$B_x(x, 0) = B_0 + \frac{1}{\mathrm{i}\,\omega}\mathcal{F}^{-1}[2\pi|k|\hat{\Delta}e]. \tag{45}$$

To demonstrate the failure of the 1-D representation of c numerically, we must choose a particular functional form; most functions with a simple maximum at $x = 0$ behave in a similar manner. We set

$$\Delta\tau(x) = \gamma\tau_0 \exp(-\beta|x|/b). \tag{46}$$

If $\gamma < 1$ this function satisfies $\|\Delta\tau\|_\infty/\tau_0 < 1$, when the iterative scheme must converge. In fact it converges for values greater than unity, because the estimates in the convergence analysis are conservative. In the following discussion we examine a model in which $\gamma = 1$ and $\beta = \frac{1}{2}$; we solve the integral equation as explained and evaluate $c(\omega)$ at $x = 0$. The locus of admittance is plotted in Fig. 5, where we see that the entire response lies outside the zone $\mathcal{Z}$. Thus the response cannot be matched by a 1-D model confined to a layer of thickness b at any frequency. Even if we lift the restriction to finite-thickness layers, the response remains incompatible with 1-D interpretation, because the phase of c is less than $-\frac{1}{2}\pi$ for the high frequency range.

Next we fit $c(\omega)$ with a model in D^+ based on 50 samples, as we did in the previous section on the TM analysis. Fig. 6 shows the

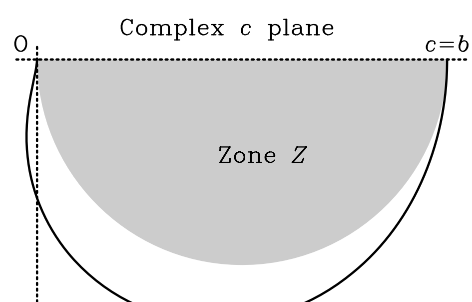

Figure 5. Locus of $c\,(\omega)$ at $x = 0$ for a sheet with conductance given by (46), and $\gamma = 1$, $\beta = \frac{1}{2}$.

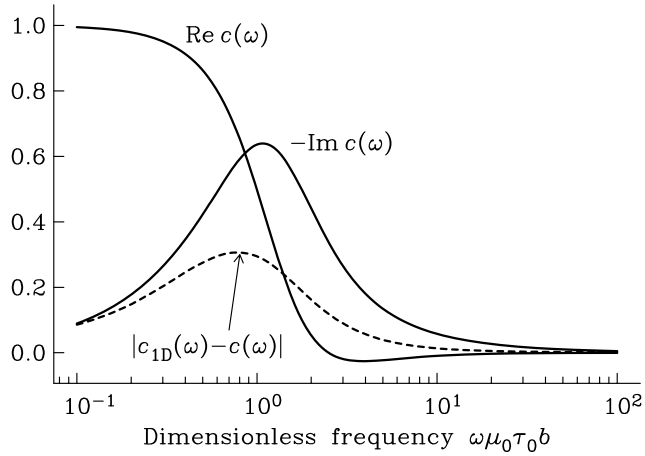

Figure 6. Admittance at $x = 0$ of the thin sheet model described by (46) with $\gamma = 1$, $\beta = \frac{1}{2}b$, together with the error of the best 1-D response fitted at 50 frequencies in the interval shown. Admittance values scaled by $1/b$.

response and the error in the best-fitting model, calculated without the constraint of a perfectly conducting base at $z = b$. The contrast with Fig. 3 is striking: the error here is four orders of magnitude larger.

4 CONCLUSIONS

We have shown by a counterexample that the MT response in the TE mode at a single site cannot always be interpreted in terms of a 1-D model under the site. Weidelt & Kaikkonen (1994) have shown this to be the case for TM induction, but the question for the TE mode has been open until now. The 1-D approximation of TM responses appears to be remarkably good in every case the author is aware of, but the example considered in this paper shows that the TE responses may be very poorly represented in this way, and therefore a quality control technique based on that approximation should be used with extreme caution, or perhaps best avoided completely.

ACKNOWLEDGMENTS

This paper is a tribute to the late Peter Weidelt, who died last year. The author, and the field of geophysical electromagnetism in general, owes Peter an enormous debt: his influence on the theory is everywhere. He was an great scholar, and a modest, generous man.

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APPENDIX A: THE ZONE $\mathcal{Z}$

We wish to prove the condition (20). In a 1-D system of thickness b, terminated by a perfect conductor, we can write the expression for the admittance in the form of eq. (2), which we repeat for convenience

$$c = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n + \mathrm{i}\,\omega}. \tag{A1}$$

Consider (A1) written in terms of its real and imaginary parts

$$c = a_0 + \sum_{n=1}^{\infty} \frac{\lambda_n a_n}{\lambda_n^2 + \omega^2} - \mathrm{i} \sum_{n=1}^{\infty} \frac{\omega a_n}{\lambda_n^2 + \omega^2}. \tag{A2}$$

Because $a_n \geq 0$ it is clear Im $c \leq 0$, which establishes the first part of (20).

Also from (A2) Re $c \geq 0$, so that we may write $c = |c|\,\mathrm{e}^{-\mathrm{i}\phi}$, where $0 \leq \phi \leq \frac{1}{2}\pi$. We will prove the second half of (20) by discovering, for a specified value of ϕ, the largest value of $|c|$ satisfying (A1), subject to the conditions that $a_n \geq 0$ and

$$a_0 + \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} = b. \tag{A3}$$

This equation is a statement that $c(0) = b$, under the assumption there is no perfect conductor within the layer.

We begin by assuming that, in addition to ϕ and ω, the λ_n are also known, but the a_n, which are non-negative, are otherwise free to be chosen to maximize the magnitude $|c|$. The fact that $\arg c = -\phi$ can be expressed as Re c + Im $c \cot\phi = 0$ which, in view of (A2), provides a condition on the a_n

$$a_0 + \sum_{n=1}^{\infty} \frac{\lambda_n - \omega\cot\phi}{\lambda_n^2 + \omega^2}\, a_n = 0. \tag{A4}$$

The quantity we wish to maximize can be written as $|c| =$ Re $c \sec\phi$, also a linear combination of the a_n

$$|c| = a_0 \sec\phi + \sum_{n=1}^{\infty} \frac{\lambda_n \sec\phi}{\lambda_n^2 + \omega^2}\, a_n. \tag{A5}$$

In this way we have set up a semi-infinite linear program to minimize a linear functional, namely, $-\,|c|$ subject to two linear constraints (A3) and (A4) over the non-negative vector $[\, a_0, a_1, a_2, a_3 \ldots]^T$.

In the terminology of classical linear programming (Gill *et al.* 1981) we know that there is a minimizing solution in the form of a *basic* vector, that is, one in which at most two elements are non-zero because there are two constraint equations. Let us assume for the moment that the two non-vanishing components are a_0 and one other, which will be a_1. Then the linear eqs (A3) and (A4) can be solved

$$a_0 = \frac{b\lambda_1}{\omega}\,\frac{\omega\cot\phi - \lambda_1}{\omega + \lambda_1\cot\phi} \tag{A6}$$

$$a_1 = \frac{b\lambda_1}{\omega}\,\frac{\omega^2 + \lambda_1^2}{\omega + \lambda_1\cot\phi}. \tag{A7}$$

Upon substituting into (A5) we find after some light algebra that

$$|c| = \frac{b\lambda_1 \operatorname{cosec}\phi}{\omega + \lambda_1\cot\phi}. \tag{A8}$$

Of all the possible values of λ_1 we can now choose the one that makes $|c|$ in (A8) as large as possible: the expression on the right is a monotone increasing function of λ_1, but because $a_0 \geq 0$, from (A6) λ_1 cannot be greater that $\omega\cot\phi$. So we must choose $\lambda_1 = \omega\cot\phi$ to maximize $|c|$; this reduces (A8) to

$$|c|_{\max} = b\cos\phi. \tag{A9}$$

Contrary to our initial assumption, the solution vector has only one positive component, but that is also permitted. If we look at the alternative type of basic solution, where $a_0 = 0$, and two other elements of the vector are non-zero, we find the maximum $|c|$ when λ_1 coalesces with λ_2, thus generating (A9) again; we omit the rather tedious algebra for this result.

Eq. (A9) is the polar equation for a circle, with diameter b, passing through the origin and the point $c = b$, in other words, the boundary of the region given by $|z - \frac{1}{2}b| \leq \frac{1}{2}b$, the second part of (20). [Q.E.D.]

What layered structure gives rise to a point on the curved boundary of $\mathcal{Z}$? Its admittance is in the form

$$c(\omega) = \frac{a_1}{\lambda_1 + \mathrm{i}\,\omega}. \tag{A10}$$

As Parker (1980) shows, this response results from a single thin sheet of conductor at the surface. If the conductance (integrated conductivity) of the sheet is τ_0, then $\lambda_1 = (\mu_0\tau_0 b)^{-1}$, and $a_1 = (\mu_0\tau_0)^{-1}$. Not coincidentally, this is the unperturbed state of the models studied in Section 3.