1. Notes on the Electromagnetic Induction within the Earth.

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1. Introduction

It is well known that the changes in earth current closely correlate with those of the earth's magnetic field. Many geophysicists hitherto endeavoured to explain this fact. Nowadays, it is established that the greatest part of the earth current is the induced electricity within the earth by the variation of the geomagnetic force of external origin.

Chapman and Whitehead[1] investigated the electromagnetic induction by the diurnal variation of the external magnetic field. They considered a model of the earth containing a uniformly conducting core surrounded by a non-conducting layer. According to them, the diurnal variation of earth current at Ebro[2], Spain, derived from the magnetic potential function, agreed roughly with the observation, while the amplitude was several times smaller.

Ertel[3], considering the heterogeneity of the electrical conductivity, extended Chapman-Whitehead's theory. The same problem in the anisotropic earth were studied by Hirayama[4] as a two-dimensional problem.

In this paper, a method to discuss the relation between the variation of geomagnetism and of earth current is described. To avoid troublesome mathematics, the writer simply treats the earth as semi-infinite media. (This assumption will be reasonable when the discussion is limited to a narrow region.)

We can obtain the intensity of magnetic or electric field as the solution of Maxwell's equation in quasi-stationary state. Taking into consideration the initial condition and the boundary condition at the surface of the earth where the magnetic field or earth current are observed, we can get the variation of the magnetic field from the observed variation of earth current or *vice versa*.

By this method, the writer discussed the diurnal variation of earth current at Ebro as Chapman and Whitehead did.

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1) S. Chapman and T. T. Whitehead, *Trans. Cambr. Phil. Soc.*, 22 (1922), 463.
2) S. Chapman and T. T. Whitehead, *Terr. Mag.*, 28 (1923), 125.
3) H. Ertel, *Veroeff. d. Preusz. Met. Inst.*, Nr. 391 (1932).
4) M. Hirayama, *Jour. Met. Soc. Japan*, 13 (1935), 456.

At first, assuming that the electrical conductivity of the earth is isotropic, the diurnal variation of earth current is calculated from the observed diurnal variation of geomagnetic field. Then, comparing the observed value to the calculated ones, the electrical conductivity of the earth is determined.

The writer further treats the same problem in the anisotropic earth. Analysing the data of earth current, in this case, the variation of geomagnetic field is calculated.

In order to avoid the accidental irregularity of the observation, the hourly values of the diurnal variation averaged through the international calm days of the year 1927 are used.

The variations ars shown in Fig. 1.

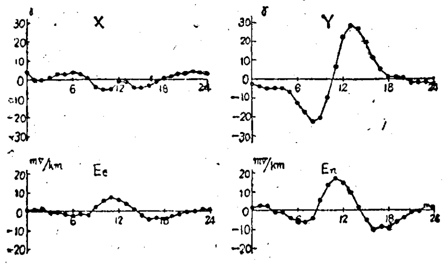

Fig. 1

Diurnal variation of the earth's magnetic field and earth current at Ebro. 1927.

2. Electromagnetic induction within semi-infinite, isotropic earth.

Electric and magnetic quantities in earth are connected by Maxwell's equations

$$\text{rot}\,\vec{H}=4\pi\,\vec{i}\,, \tag{1}$$

$$\text{rot}\,\vec{E}=-\frac{\partial\vec{B}}{\partial t} \tag{2}$$

for the phenomena are quasi-stationary, where $\vec{H}$, $\vec{B}$, $\vec{E}$ and $\vec{i}$ denote magnetic field, magnetic induction, electric field and electric current-density respectively. Between these quantities we have the next relations,

$$\vec{i}=\sigma\vec{E}\,, \tag{3}$$

$$\vec{B}=\mu\vec{H}\,, \tag{4}$$

and besides

$$\text{div}\,\vec{B}=0 \tag{5}$$

where σ and μ are specific electrical conductivity and magnetic permeability of earth respectively. In this section, it is assumed that the earth is isotropic and homogeneous, so these quantities are both scalar and constant everywhere.

Eliminating $\vec{E}$ from these equations, we have

$$\frac{\partial\vec{H}}{\partial t}=\kappa^2\nabla^2\vec{H} \tag{6}$$

where

$$\kappa^2 = 1/4\pi\sigma\mu. \tag{7}$$

In our problem, as the constant part of the earth's magnetic field does not contribute to the variation of electric field in the earth, so we can take the next relation as the initial condition without loss of generality,

$$\text{at } t=0, \quad \vec{H}=0 \tag{8}$$

and the boundary condition is

$$\text{at the surface } (z=0),\ \vec{H}=\vec{H}_0(t) \tag{9}$$

while the axes x, y and z are taken to be north-, east-, and downwards respectively.

Then the problem becomes equivalent to the typical one in one-dimensional heat conduction in a semi-infinite solid whose solution can be expressed in the well-known form

$$\vec{H} = \frac{2}{\sqrt{\pi}} \int_{\frac{z}{2\kappa\sqrt{t}}}^{\infty} e^{-\beta^2} \vec{H}_0\left(t - \frac{z^2}{4\kappa^2\beta^2}\right) d\beta$$ [5]

When the magnetic field at the surface can be expressed by such a series of harmonic functions as

$$\vec{H}_0 = \sum_{s=1}^{n-1} \vec{a}_s \sin spt, \tag{11}$$

the magnetic field in the earth becomes

$$\vec{H} = \sum_{s=1}^{n-1} \vec{a}_s e^{-\frac{z}{\kappa}\sqrt{\frac{sp}{2}}} \sin\left(spt - \frac{z}{\kappa}\sqrt{\frac{sp}{2}}\right) - \frac{2}{\sqrt{\pi}} \sum_{s=1}^{n-1} \vec{a}_s \int_0^{\frac{z}{2\kappa\sqrt{t}}} e^{-\beta^2} \sin\left(spt - \frac{spz^2}{4\kappa^2\beta^2}\right) d\beta. \tag{12}$$

Then, considering the equation (2), we can get the intensity of the electric field from the above equation. If we assume that $\vec{H}$ does not depend both on x and y, the vertical component of the electric field vanishes everywhere, while the north and east components at the surface become

$$\left.\begin{aligned} E_{0x} &= \sqrt{\frac{\mu p}{4\pi\sigma}} \sum_{s=1}^{n-1} \sqrt{s}\, a_{sy} \sin\left(spt + \frac{\pi}{4}\right), \\ E_{0y} &= \sqrt{\frac{\mu p}{4\pi\sigma}} \sum_{s=1}^{n-1} \sqrt{s}\, a_{sx} \sin\left(spt + \frac{\pi}{4}\right). \end{aligned}\right\} \tag{13}$$

When the hourly values of the diurnal variation of the north and east

5) For example, H. S. Carlslaw, *Introduction to the mathematical theory of heat in solids*. 2 nd edition, (1921), p. 47.

components of the earth's magnetic field are given, we can obtain the numerical values of the coefficients $a_{1x}, \cdots, a_{1y} \cdots$ by means of harmonic analysis. Then, combining these values, we can calculate the series of the right-hand side of (13) and so, considering the hourly values of the earth potential, the most probable values of $\sqrt{\frac{\mu p}{4\pi\sigma}}$ will be determined by means of least squares. Since $p=\frac{\pi}{24\times60\times60}$ and practically $\mu=1$, then, we can determine the most probable values of σ.

In this article, as it is a sumed that the earth is electrically isotropic, specific values of the electrical conductivity can be determined independently from either combinations of the east component of geomagnetic fo ce and the north component of earth current or the ones of north component of geomagnetic force and the east component of earth current, denoting the former σ_x and the latter σ_y respectively.

At Ebro, the amplitude of the east component is about three times larger than he north component in geomagnetic diurnal variation and *vice versa* in earth current. The determined value σ_y, therefore, will be more reliable than σ_x for the disturbance in earth current, which will be caused by some other origins such as contact electromotive forces at the surfaces of the electrodes, will affect equally to the diurnal variation of ea th current. In 1927, the determined value of σ_y and σ_x, expressed in terms of 10^{-6} ohm^{-1} cm^{-1}, is 4.7 ± 0.9 and 3.3 ± 1.8, whe e the probable errors are comparatively large because the errors in $\sqrt{\frac{\mu p}{4\pi\sigma}}$ become twice in σ.

In the next place, we can obtain the diurnal variation of earth current using the above calculated values of the electrical conductivity of the earth, as shown in Fig. 2. The calculated figures roughly ag ee with the observation.

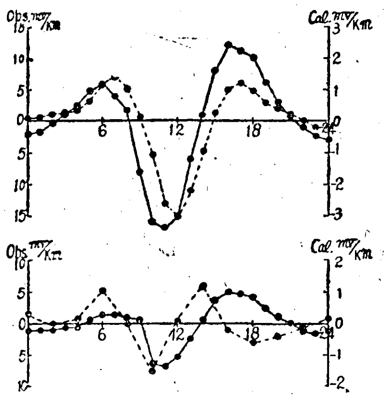

Fig. 2
Upper Figures—Diurnal Variation of the N Gradient of Earth Potential, 1927.
Lower Figures—Diurnal Variation of the E Gradient of Earth Potential, 1927.
(Full and broken lines corespond respectively to the observation and the calculation).

About twenty years ago, Chapman and Whitehead[6] computed theoretically the diurnal variation of earth current at Ebro. They determined several coefficients of

6) Chapman and Whitehead, *loc. cit*.

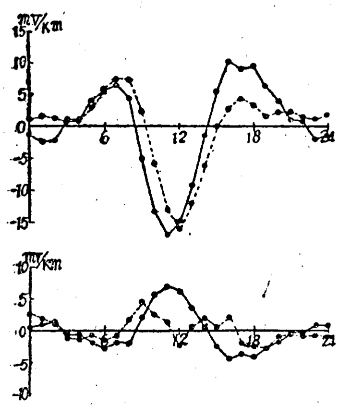

Fig. 3
Upper Figures—Diurnal Variation of the N Gradient of Earth Potential.
Lower Figures—Diurnal Variation of the W Gradient of Earth Potential.
(Full and broken lines correspond respectively to the observation and the calculation)
(After Chapman and Whitehead)

s herical harmonic expa sion of geomagnetic potential by analysing the geomagnetic data (averaged diurnal variaion from 1914 to 1918) observed at several stations distributed all over the world, and then computed the diurnal variation of earth current on the stand-point of electromagnetic induction theory. The diurnal variations obtained by their investiia-tions are shown in Fig. 3. The figures well resemble with those in F.g. 2. As the annual means of geomagnetic forces do not extraordinarily differ every year, it is reasonab e that we obtain similar conclusions with their invetsigation, when the earth can be treated as semi-infinite media. In F g. 2, there are somewhat systematic differences between the observed figüres and the computed ones as well as in Fig. 3.

If we eliminate $\vec{H}$ in Maxwell's equation, the process of the above mentioned theory will be also available for the discussion of the variation of the magnetic field from the observed variation of earth current.

3. Electromagnetic induction within semi-infinite, anisotropic earth.

It has been often reported[7)8)9)] that the variation of earth cur ent does not occur in the direction pe pendicular to that of the earth's magnetic field. Consequently, there must exist apparent anisotropy of the electrical conductivity of earth. This anisotropy is observed even by direct measurements[10)11)].

Now we shall extend the electromagnetic induction theory to the anisotropic earth.

In this case, Ohm's law becomes

7) H. Hatakeyama, *Geoyhys. Mag.*, 12 (1938), 1891.
8) T. Nagata, *Proc. Imp. Acad. Tokyo*, 20 (1944), 81.
9) T. Nagata, *Bull. Earthq. Res. Inst.*, 22 (1944), 72.
10) T. Nagata, Read at the Nov. Meeting of the Earthq. Res. Inst., (1945).
11) K. Hirao, Read at the Mar. Meeting of the Earthq. Res. Inst., (1946).

$$\vec{i}=(\sigma)\vec{E} \tag{14}$$

where ( ) means tensor.

Eliminating $\vec{H}$ from Maxwell's equation, we get

$$4\pi(\sigma)\mu\frac{\partial\vec{E}}{\partial t}=\nabla^2\vec{E}-\text{grad div}\,\vec{E} \tag{15}$$

If we assume tnat one of the directions of the principal axes coincides with the vertical line to the surface, each component of electric field must satisfy the next equations

$$\left.\begin{aligned}&\frac{\partial E_x}{\partial t}=\kappa_x^2\frac{\partial^2 E_x}{\partial z^2} \qquad (\kappa_x^2=1/4\pi\sigma_{xx}\mu)\\&\frac{\partial E_y}{\partial t}=\kappa_y^2\frac{\partial^2 E_y}{\partial z^2} \qquad (\kappa_y^2=1/4\pi\sigma_{yy}\mu)\\&\frac{\partial E_z}{\partial t}=0\end{aligned}\right\} \tag{16}$$

where x, y and z are the directions of the principal axes.

The initial and boundary conditions are

$$\left.\begin{aligned}&\text{at } t=0 \qquad \vec{E}=0\ ,\\&\text{at the surface} \qquad \vec{E}=\vec{E}_0(t)\end{aligned}\right\} \tag{17}$$

as well as in the former article.

When the electric field at the surface can be expressed by a series of harmonic functions auch as

$$\vec{E}_0=\sum_{s=1}^{n-1}\vec{A}_s\sin spt\ , \tag{18}$$

the electric field in the earth becomes

$$\left.\begin{aligned}E_x&=\sum_{s=1}^{n-1}A_{xs}e^{-\frac{z}{\kappa_x}\sqrt{\frac{sp}{2}}}\sin\left(spt-\frac{z}{\kappa_x}\sqrt{\frac{sp}{2}}\right)-\frac{2}{\sqrt{\pi}}\sum_{s=1}^{n-1}A_{xs}\int_0^{\frac{z}{2\kappa_x\sqrt{t}}}e^{-\beta^2}\sin\left(spt-\frac{spz^2}{4\kappa_x^2\beta^2}\right)d\beta\ ,\\E_y&=\sum_{s=1}^{n-1}A_{ys}e^{-\frac{z}{\kappa_y}\sqrt{\frac{sp}{2}}}\sin\left(spt-\frac{z}{\kappa_y}\sqrt{\frac{sp}{2}}\right)-\frac{2}{\sqrt{\pi}}\sum_{s=1}^{n-1}A_{ys}\int_0^{\frac{z}{2\kappa_y\sqrt{t}}}e^{-\beta^2}\sin\left(spt-\frac{spz^2}{4\kappa_y^2\beta^2}\right)d\beta\ .\end{aligned}\right\} \tag{19}$$

Substituting these relations into rot $\vec{E}=-\partial\vec{B}/\partial t$ and integrating with respect to time, we get the magnetic field. At the surface, the obtained magnetic field is written as follows;

$$\left.\begin{aligned}H_{0x}&=\frac{1}{\mu\kappa_y}\sum_{s=1}^{n-1}\frac{A_{ys}}{\sqrt{sp}}\cos\left(sp+\frac{\pi}{4}\right),\\H_{0y}&=-\frac{1}{\mu\kappa}\sum_{s=1}^{n-1}\frac{A_{xs}}{\sqrt{sp}}\cos\left(spt+\frac{\pi}{4}\right).\end{aligned}\right\} \tag{20}$$

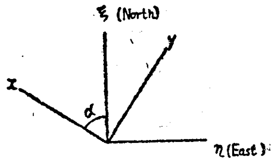

Fig. 4

W en the angle between x- and ξ- (north) direction is α, as appears in Fig. 4, we can write the north and east components of the magnetic field at the surface in the next expressions;

$$
\left.
\begin{aligned}
H_{0\xi} &= \frac{1}{\mu}\left(\frac{1}{\kappa_y}-\frac{1}{\kappa_x}\right)\frac{\sin 2\alpha}{2}\sum_{s=1}^{n-1}\frac{A_{\xi s}}{\sqrt{sp}}\cos\left(spt+\frac{\pi}{4}\right) \\
&\quad +\left\{\frac{1}{2}\left(\frac{1}{\kappa_y}+\frac{1}{\kappa_x}\right)+\frac{\cos 2\alpha}{2}\left(\frac{1}{\kappa_y}-\frac{1}{\kappa_x}\right)\right\}\sum_{s=1}^{n-1}\frac{A_{\eta s}}{\sqrt{sp}}\cos\left(spt+\frac{\pi}{4}\right), \\
H_{0\eta} &= -\frac{1}{\mu}\left\{\frac{1}{2}\left(\frac{1}{\kappa_y}+\frac{1}{\kappa_x}\right)-\frac{\cos 2\alpha}{2}\left(\frac{1}{\kappa_y}-\frac{1}{\kappa_x}\right)\right\}\sum_{s=1}^{n-1}\frac{A_{\xi s}}{\sqrt{sp}}\cos\left(spt+\frac{\pi}{4}\right) \\
&\quad -\frac{1}{\mu}\left(\frac{1}{\kappa_y}-\frac{1}{\kappa_x}\right)\frac{\sin 2\alpha}{2}\sum_{s=1}^{n-1}\frac{A_{\eta s}}{\sqrt{sp}}\cos\left(spt+\frac{\pi}{4}\right).
\end{aligned}
\right\}
\tag{21}
$$

where

$$
\sum_{s=1}^{n-1} A_{\xi s}\sin spt = E_{0\xi}, \quad \sum_{s=1}^{n-1} A_{\eta s}\sin spt = E_{0\eta}. \tag{22}
$$

Then, in the same manner with those of the isotropic case, we can determine α, κ_x and κ_y or consequently σ_{xx} and σ_{yy}.

As a result of the actual determination, we get from the data of 1927

direction of minimum σ	N 18° W
σ_{min}	$0.32\times10^{-6}\,\text{ohm}^{-1}\,\text{cm}^{-1}$,
σ_{max}	$2.0\times10^{-6}\,\text{ohm}^{-1}\,\text{cm}^{-1}$,
$\sigma_{min}/\sigma_{max}$	0.16.

Using these values, the diurnal variation of the earth's magnetic field expected from the theory is shown in Fig. 5. The observation agrees roughly with the calculation.

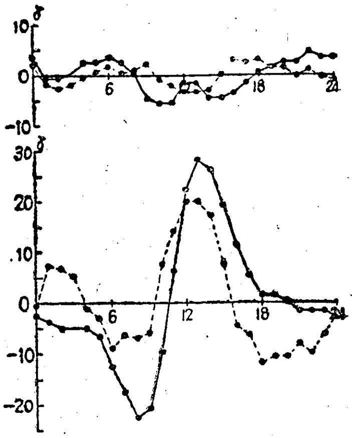

Fig. 5
Upper Figures—Diurnal variation of X, 1927.
Lower Figures—Diurnal variation of Y, 1927.
(Full and broken lines correspond respectively to the observation and the calculation)

In the present case, the procedure of the determination of the electrical conductivity is much more complicated than in the isotropic case. For this reason, the obtained values will be less reliable. In both cases, the determined values of the electrical conductivity agree in its order.

As appear in Fig. 6, the direction of variation of earth current is nearly N 20° W, while that of magnetic field is E-W. When the earth is isotropic, the induced currents flow to the direction perpendicular to the geomagnetic variation. Therefore, the present deviation of the direction of earth current will be due to such anisotropy of the electrical conductivity as the direction of minimum σ=N 18° W as was given by the theory mentioned above. It is reasonable that the earth potential predominates in the direction of high resistance. The determination will become more accurate if we use magnetic variations (such as "bay-type" disturbances) whose directions woud be nearly parallel to that of minimum σ.

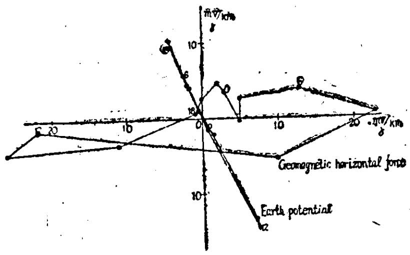

Fig. 6
Vector diagram of diurnal variation of the geomagnetic horizontal force and the earth current.

4. Conclusion.

A practical method to discuss the relation between the variation of the earth's magnetic field and that of earth current was described in this paper both in isotropic and anisotropic earth.

Although the writer assumes that the earth is semi-infinite media, the earth current calculated from the observed variation of geomagnetic field agree fairly well with the one derived from the spherical harmonic analysis of the magnetic potential investigated by Chapman and Whitehead who used the world-wide data.

The anisotropy of the electrical conductivity agrees with the fact that the direction of the diurnal variation of earth current at Ebro is nearly N 20° W because the calculated direction of minimum σ is N 18° W.

Unfortunately, the determination of the electrical conductivity is rather rough for our method is not a direct measurement. For this reason, it seems meaningless to discuss the time-change of the electrical conductivity and its anisotropy by this method. It is desirable to measure these quantities directly and detect their change during a comparatively long period.

Anyhow, the present study shows that there exist some anisotropy in the electrical property of earth as was often reported. About the cause of this anisotropy we have not yet precise knowledge. In strained state earth may show some anisotropy, and besides certain heterogenity near the point where the observations are executed may also be a probable cause of the anisotropy.

It will be of interest and importance to investigate the relation between the anisotropy and the mechanical structure of the earth's crust.

The writer wishes to express his sincere thanks to Prof. F. Kishinouye for his advices in the course of this study.

1. 地磁氣變化に伴う大地中の電磁感應

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地電流の大部分が地球外部に原因を有する地球磁場の變化に伴つて誘導されるものであることは多くの地球物理學者に依つて確められている.

本報文に於ては地表に於て觀測された地磁氣若は地電位差變化の値から，地電位差若は地磁氣の變化を理論的に求める一方法を提出し，實測と計算との比較から土地の電氣傳導度を求め得ることを示した.

はじめに大地は電氣的に等方であるとして取扱つた. この場合は互に直角な地磁氣及び地電位差の變化を比較することにより，土地の電氣傳導度を求めることが出來る. 例として Spain の Ebro の日變化の一年間の平均に就て實際に此の方法を適用してみたが，地磁氣變化より求めた地電位差變化は，Cahpman と Whitehead が地磁氣ポテンシャルの球函數展開から求めたものと形に於て非常によく一致した.

次に最近問題になつている土地の異方性を考慮にいれた場合を取扱つた. この場合は Maxwell の方程式を解く時の便宜上，地電位差の變化をもととして如何なる地磁氣の變化が期待されるかを調べた. 上述の Ebro の日變化を例として解析を行つた結果は，電氣傳導度最小の方向は N18°W 又最大値と最小値の比は 0.2 程度となつた. このことは Ebro の地電流の變化が殆ど N20°W の方向に限られてゐることをよく說明するものと云えよう. この場合地磁氣の變化は水平ヴェクトル圖に於て略〻東西であるから，異方性を求める上にはあまり適していない. 例えば地電位差變化の方向とほゞ平行に近いような地磁氣の變化を用いて求めたならば，更に精度よく異方性を求めることが出來よう.

いずれにしても此方法は間接的であつて種々の誤差を含み易いし，精度も不充分であるから，土地の電氣的性質の時間的變化を調べるには不適であつて，直接測定を實施することが望ましい.

Eb o の例に見る如く，土地の電氣的性質は少くとも見掛け上異方性を有する. この異方性を生ずる原因また他の地球物理學的現象特に地殼の力學的性質との關係を調べることは，極めて興味あることゝ考えられ，今後の研究に期待する處大である.