ON THE RELATION BETWEEN TELLURIC CURRENTS
AND THE EARTH'S MAGNETIC FIELD*

JAMES R. WAIT†

ABSTRACT

The validity of Cagniard's analysis of the behaviour of telluric earth currents is questioned in view of the fact that the harmonic components of the electric field and the magnetic field tangential to the ground are only proportional to one another if the fields are sufficiently slowly varying over the surface of the ground. His result is extended to include the effects of a layered ground with both conductivity and susceptibility variations. Finally the corresponding transient problem is solved for a two-layer horizontally stratified earth.

In a recent interesting and important paper by Cagniard (1953), the relation between telluric earth currents and variation of the earth's magnetic field is studied. In the present paper the validity of his results is discussed and several extensions to the theory are pointed out.

The impedance concept in electromagnetism, as first disclosed so lucidly by Schelkunoff (1938), is well suited to this type of problem. If the electric and magnetic forces tangential to a surface S at a point P are mutually perpendicular, the ratio of the tangential electric force to the tangential magnetic force is termed the field impedance normal to S at P. In general the tangential electric and magnetic forces are not perpendicular so that the field impedance does not exist; even if it does exist, it may depend upon the system of sources by which the fields have been set up (Monteath, 1951).

Let the x, y plane be the boundary between two semi-infinite homogeneous media, the fields being set up by sources in the upper medium. It can be shown that in this plane the tangential electric fields E_x and E_y are related to the tangential magnetic fields by

$$E_x = \eta H_y + \frac{\eta}{2\gamma^2}\left(\frac{\partial^2 H_y}{\partial y^2} - \frac{\partial^2 H_y}{\partial x^2} + 2\frac{\partial^2 H_x}{\partial x \partial y}\right) + \text{terms in } \gamma^{-4}, \text{ etc.}$$

and

$$-E_y = \eta H_x + \frac{\eta}{2\gamma^2}\left(\frac{\partial^2 H_x}{\partial x^2} - \frac{\partial^2 H_x}{\partial y^2} + 2\frac{\partial^2 H_y}{\partial y \partial x}\right) + \text{terms in } \gamma^{-4}, \text{ etc.} \tag{1}$$

where, $\gamma = i\sigma\mu\omega - \epsilon\mu\omega^2$ is the intrinsic propagation constant of the lower medium and $\eta = i\mu\omega/\gamma$ is its intrinsic characteristic impedance. The constants σ, ϵ, μ are the conductivity, dielectric constant, and permeability of the lower medium.

Equation (1) was obtained by regarding the distribution of H_x and H_y over

---

the x, y plane as an aperture distribution giving rise to an angular spectrum of plane waves (Monteath, 1951). If H_x and H_y vary sufficiently slowly over the x, y plane, or if γ is sufficiently large, only the first term in each of the formulae need be retained. For example, if the frequency is 0.1 cycle (a period of 10 sec) and the conductivity of the ground is 10^{-1} mho/meter then $\left| 1/\gamma \right| = 35$ kilometers. Then for this case if H_x and H_y do not change greatly between points 35 km apart, the terms in γ^{-2}, etc. may be neglected and equation (1) becomes:

$$E_x \approx \eta H_y$$
$$E_y \approx - \eta H_x. \tag{2}$$

Thus the field impedance normal to the boundary exists and is equal to the intrinsic impedance of the lower medium. For periods longer than about 10 seconds it is probable that H_x and H_y vary by an appreciable amount in the distance corresponding to $\left| \gamma^{-1} \right|$ since it is believed that the sources of the variations of the earth's magnetic field are at heights of the order of 100 kilometres as evidenced from rocket measurements (Mitra, 1952, p. 574; Baker and Martyn, 1952). This would vary considerably during periods of magnetic activity.

Equation (2) is exact if the sources in the upper medium (air) give rise to plane waves incident normally on the lower medium (ground). For most cases the sources in the atmosphere do not give rise to normally incident plane waves. Cagniard recognizes this fact and also arrives eventually at equations exactly equivalent to equation (2). He does not recognize, however, that the plane wave spectrum can contain plane waves with a complex angle of incidence α. He states that $\sin^2 \alpha$ is never greater than unity in magnitude, which is of course incorrect. The limitation that H_x and H_y should not vary greatly in a distance $\left| \gamma^{-1} \right|$ is therefore not evident from Cagniard's analysis.

When the ground is horizontally stratified, consisting of say an upper stratum of thickness h with an intrinsic propagation constant γ_1 and intrinsic impedance η_1 and a lower stratum of infinite thickness with constants γ_2 and η_2, the surface impedance is given by

$$\eta = \eta_1 \frac{\eta_2 + \eta_1 \tanh (\gamma_1 h)}{\eta_1 + \eta_2 \tanh (\gamma_1 h)} . \tag{3}$$

This formula can be arrived at directly by considering the analogous transmission line problem (Schelkunoff, 1938). The equivalent line is of length h with a propagation constant γ_1 and characteristic impedance η_1 and is terminated in an impedance whose value is η_2. The input impedance of the line is then η as given by equation (3), which can now be written in terms of the conductivities σ_1 and σ_2, and magnetic permeabilities μ_1 and μ_2 of the upper and lower strata as follows:

$$\eta = \eta_1 Q \tag{4}$$

where

$$Q = \frac{(\mu_2\sigma_1/\mu_1\sigma_2)^{1/2} + \tanh (i\sigma_1\mu_1\omega)^{1/2}h}{1 + (\mu_2\sigma_1/\mu_1\sigma_2)^{1/2} \tanh (i\sigma_1\mu_1\omega)^{1/2}h}$$

and it has been assumed that $\epsilon_1\omega \ll \sigma_1$ and $\epsilon_2\omega \ll \sigma_2$. This equation now reduces to Cagniard's equation (30) for the special case when the permeability contrast between the strata is zero (i.e. $\mu_1 = \mu_2 = \mu_0$). It is therefore pointed out that for cases of formations having magnetic susceptibility, ratios σ_2/σ_1, given in his Figures 7 and 8, should be replaced by the ratio $\mu_1\sigma_2/\mu_2\sigma_1$, and the horizontal scale (inverse frequency) should be shifted to the left by a factor μ_2/μ_1.

To apply Cagniard's results to interpretation of sub-surface stratification it is necessary to carry out a harmonic analysis of both the electric field function $e_x(t)$ and the magnetic field function $h_y(t)$. The harmonic components $E_x(\omega)$ and $H_y(\omega)$ at an angular frequency ω are then compared to obtain the surface impedance function $\eta(\omega)$. Apparently this procedure requires a fair amount of labor unless the analysis be carried out by electronic means. A suggested alternative is to examine the relation between the time functions $e_x(t)$ and $h_y(t)$ directly.

If the magnetic field $h_y(t)$ is in the form of a step-function, that is, it suddenly changes from one level H_0 to another level $H_0 + \Delta H_0$ at $t = 0$ then the frequency spectrum $H_y(\omega)$ associated with this change is

$$H_y(\omega) = \int_{-\infty}^{+\infty} h_y(t)e^{-i\omega t}dt = \int_0^\infty \Delta H_0 e^{-i\omega t}dt = \frac{\Delta H_0}{i\omega} . \tag{5}$$

The corresponding frequency spectrum of the tangential electric field is then given by:

$$E_x(\omega) = \eta(\omega)\Delta H_0/i\omega .$$

The frequency function $\eta(\omega)$ is given by equation (3) for the two-layer ground and it can be rewritten in the following form.

$$\eta(\omega) = \frac{i\mu_1\omega}{\gamma_1}\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{1-\beta}{1+\beta}\right)^n e^{-2n\gamma_1 h}\right] \tag{6}$$

where $\beta = (\mu_1\sigma_2/\mu_2\sigma_1)^{1/2}$. The time function $e_x(t)$ for the tangential electric field is then given by:

$$e_x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_x(\omega)e^{-i\omega t}d\omega$$

$$= \frac{1}{2\pi}\left(\frac{\mu}{\sigma}\right)^{1/2}\Delta H_0 \int_{-\infty}^{+\infty}\left[\frac{1}{(i\omega)^{1/2}} + 2\sum_1^{\infty}\left(\frac{1-\beta}{1+\beta}\right)^n \frac{e^{-2n\alpha(i\omega)^{1/2}}}{(i\omega)^{1/2}}\right]e^{i\omega t}d\omega. \tag{7}$$

These integrals are of a known type (Campbell and Foster, 1948, p. 93, formula no. 807) so the integrations can be readily carried out to yield

$$e_x(t) = F(T)/\pi^{1/2}\sigma h$$

where

$$F(T) = T^{-1/2} \sum_{n=1}^{\infty} \left[1 + 2\left(\frac{1 - \beta}{1 + \beta}\right)^n e^{-n^2/T} \right] \tag{8}$$

and

$$T = t/\alpha^2 = t/\sigma_1\mu_1 h^2.$$

The function $F(T)$ is therefore characteristic of the shape of the electric field time function. It is plotted in Figure 1 as a function of the normalized time factor T. The case where $\beta = 1$ corresponds, of course, to the homogeneous ground where the electric field decays as $T^{-1/2}$. It is noted that for highly conducting lower layers $e_x(t)$ decays more rapidly than for the homogeneous ground. Conversely, if the lower layer is very poorly conducting the decay is somewhat slower.

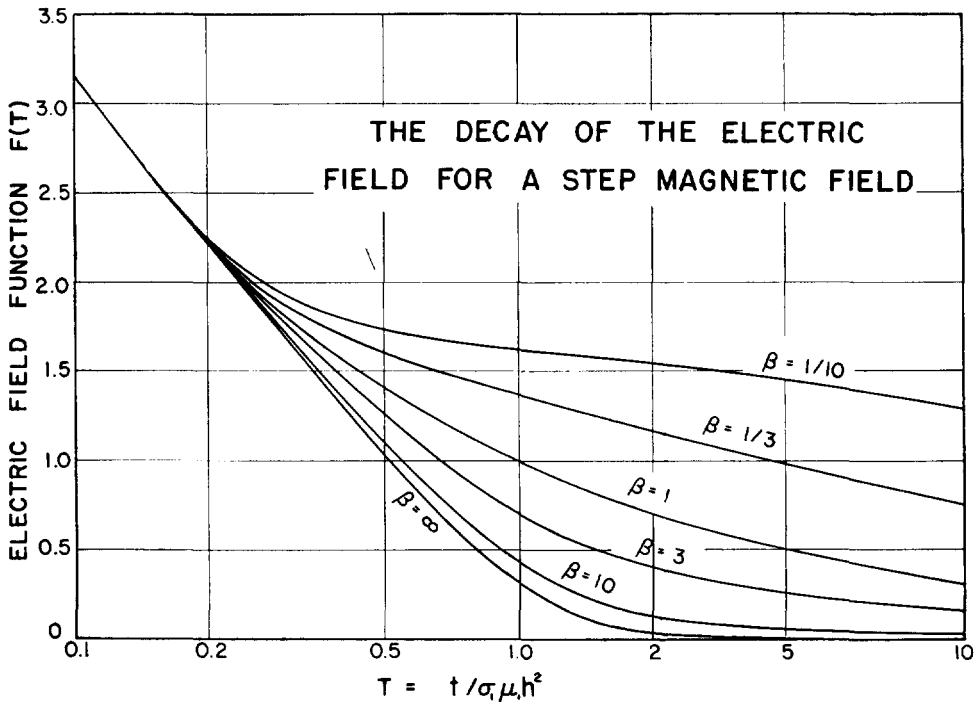The corresponding electric field function $e_x(t)$ for any other type of magnetic

FIG. 1. Electric field variation with time for two-layered earth when magnetic field varies step-wise at $t = 0$. The parameter $\beta = (\mu_1\sigma_2/\mu_2\sigma_1)^{1/2}$ where σ_1 and μ_1 are conductivity and magnetic permeability, respectively, of the upper layer, σ_2 and μ_2 of the lower layer.

field function $H_y(t)$ can be obtained in a similar manner. For example if the magnetic field is suddenly decreased in value by an amount ΔH_0 at time $t = 0$ and then is re-established at $t = t_1$, the corresponding electric field $e_{x_1}(t)$ is given by

$$e_{x_1}(t) = e_x(t - t_1) - e_x(t)$$

where $e_x(t)$ is a step-function response given by equation (8).

For a magnetic field function $\bar{h}(t)$ given by

$$\bar{h}(t) = H_0 + \Delta H_0 f(t) \quad \text{for} \quad t > 0$$
$$= H_0 \quad \text{for} \quad t < 0$$

then the corresponding electric field function $\bar{e}_x(t)$ is expressed in terms of the step-function response $e_x(t)$ by:

$$\bar{e}_x(t) = \frac{d}{dt} \int_0^t f(t - \tau) e_x(\tau) d\tau.$$

The limitation that H_x (and H_y) varies slowly over the x, y plane also applies to the transient case. An equivalent statement is that H_x should not vary appreciably in a distance $(t/2\pi\sigma\mu)^{1/2}$. This means that the transient curves for $e_x(t)$ are probably not valid for any value of t greater than about 10 seconds.

I would like to thank Mr. D. A. Trumpler for his assistance with the calculations for Figure 1.

<div align="center">REFERENCES</div>

Baker, W. G. and Martyn, D. F., 1952, The conductivity of the ionosphere: Nature, v. 170, p. 1090·

Cagniard, Louis, 1953, Basic theory of the magneto-telluric method of geophysical prospecting: Geophysics, v. 18, p. 605–635.

Campbell, G. A. and Foster, R. M., 1948, Fourier integrals for practical application: New York, D. Van Nostrand Co., Inc.

Mitra, S. K., 1952, The upper atmosphere: The Asiatic Society.

Monteath, G. D., 1951, The application of the compensation theorem to certain radiation and propagation problems: Proc. Inst. Elect. Eng., Pt. IV, v. 98, no. 1.

Schelkunoff, S. A., 1938, The impedance concept and its application to problems of reflection, refraction, shielding, and power absorption: Bell System Tech. Jour., v. 17, p. 17.

<div align="center">CORRESPONDENCE BETWEEN PROF. CAGNIARD AND DR. WAIT</div>

Before submitting the manuscript of the paper above to GEOPHYSICS, Dr. Wait sent a copy of it to Prof. Cagniard in France. The correspondence between them that followed has been made available by its authors for publication along with the paper and the letters that were exchanged are presented herewith. Acknowledgment is made to Dr. Robert G. Van Nostrand for translating Prof. Cagniard's letters.—THE EDITOR

Paris
24 October, 1953

Dr. James R. Wait
Defence Research Board
Ottawa, Canada

Dear Sir:

I have only recently received your interesting work for which I thank you very much. I assure you that I have not yet had the necessary time to study carefully its contents nor to refer to the bibliographic references which you cite. However, I do not want to delay giving you my first impressions.

With respect to transients, I stated (Geophysics, XVIII, p. 611) that I did not believe that they present great practical interest. It is to be understood that it would please me if you were to find a convenient and effective interpretive method based upon them. But, until I can be convinced of a better technique, I prefer the harmonic analysis and the use of the notion of apparent resistivity. Do not reproach me for not yet having told everything nor for not yet having explained the means by which I envisage making automatically the analysis in question. I am not in a position at present to furnish the details of the apparatus which I use to this end.

My calculations depend on the dual hypothesis of a uniform telluric sheet and a horizontally stratified geologic structure. They are, of course, only approximations. The merit of your work, merit which I appreciate greatly, is to furnish a critique of the value of this approximation. However, I would be very happy if you would show me how you establish equation (1) so that I can judge the hypothesis upon which it is founded. It would be equally agreeable to me if you would publish that at some time or another.

I would also like to make two remarks to you:

1. In the example which I have taken of a plane incident wave, I had no other aim than to show how very diverse causes could give place to the flow of the same type of current sheet in the earth. Since I considered only real plane waves, waves of the physicist and not those of the mathematician, my angles were real and I was not "incorrect" in stating that the absolute value of their sines was less than 1. In your case, you chose to consider a source situated at a finite distance and, following Weyl, to decompose the dipole emission into plane waves which are only mathematical abstractions and for which arise complex angles. I do not reproach you (for this approach) but do not criticize me for what I am doing; we are not treating the same problem.

2. Your argument concerning the lack of uniformity of telluric sheets seems to me to be open to criticism on the ground that magneto-telluric perturbations found at a given place are certainly not emitted by some isolated dipole situated in the ionosphere near the zenith. One would not explain then all of the remarkable uniformity of telluric sheets, a uniformity which is not simply a point of view nor a more or less adventuresome hypothesis but a proven fact of daily experience. When we record simultaneously in the Paris region and in the Midi of France, at two stations separated by 600 kilometers, over geologic structures which have absolutely nothing in common, rapid variation (a few seconds to a few minutes) of the horizontal components of the magnetic field, the two graphs obtained are striking in their near identity. Telluric recordings made simultaneously in France, Morocco, and Madagascar are certainly not identical—which would be contradictory—but never the less present an indisputable "family resemblance." All of this proves, if it is necessary, that magneto-telluric perturbations are not generated by isolated dipoles but by vast systems of ionospheric electric currents whose dimensions are on a global scale. The altitude of these sheets of ionospheric current can very well be of the order of only ten kilometers without introducing very much distortion in the uniformity of the telluric sheet over distances of several tens of kilometers.

I am sending a copy of this letter to Dr. Milton Dobrin. I thank you again for the interest that

you have shown in my work and the useful additions which you have made to it. Until I hear from you again, please accept, dear sir, my best wishes.

L. CAGNIARD
397, Rue Vaugirard
Paris, France.

Defence Research Telecommunications
Establishment,
Radio Physics Laboratory,
Defence Research Board,
Shirley Bay,
Ottawa, Ontario.
October 29, 1953

Professor Louis Cagniard,
397 Rue de Vaugirard,
Paris 15,
France.

Dear Professor Cagniard:

Thank you for comments on my note entitled "On the Relation Between Telluric Currents and the Earth's Magnetic Field." I am glad that you found this of interest.

I agree that the harmonic analysis of the earth current and the tangential magnetic field records would lead to a more refined method of interpretation if the Fourier components could be extracted by electronic means. In any case the transient and the frequency response data are equivalent from a mathematical standpoint. In addition, I believe that the "raw data" from the field records could be examined directly and compared to the idealized transient curves to ascertain quickly the general nature of the crustal layers.

I don't believe there is any doubt as to the existence of complex angles of incidence in the plane wave spectrum of radiation from a dipolar source. This question has been discussed at length by Booker and Clemmow.[1] Of course at large distances from the dipole the wave front is almost plane and the angle of incidence corresponds to the angle between the line of sight and the vertical. At shorter distances the field of the dipole becomes very different. The E and H fields are no longer mutually perpendicular to the radial direction and they are not related by the characteristic impedance of free space. In this case the spectrum in terms of plane waves contains complex angles of incidence so that $\sin^2 \alpha$ is not always less than unity. Of course, if the source were a current sheet of infinite extent and uniform in amplitude and phase, the emanating radiation, obtained by integrating over all the horizontal electric dipoles, would be a normally incident plane wave.

The current system in the upper atmosphere giving rise to short period (less than 60 sec) variations of the magnetic such as "micro pulsations" are confined to a limited region as observed by several investigators.[2,3,4] In this case the infinite sheet representation is not permissible. It would seem reasonable that the currents could be represented by a finite distribution of electric and magnetic dipoles and therefore one must admit complex angles of incidence. On the other hand there have been short period (5 to 10 sec) pulsations of small amplitude reported,[5] that do occur simultaneously over large parts of the world at lower and middle latitudes. At northern latitudes, however, the short period variations are generally due to local conditions in the ionosphere. In any case, it would

always be necessary to ensure that the magnetic field components do not change appreciably in a distance equal to the reciprocal of the propagation constant of the ground.

At your request I shall outline briefly the derivation[6] of equation (1) in my note.

A rectangular coordinate system is employed with the (x, y) plane coinciding with the ground and the z axis pointing downward. The ground is assumed to be homogeneous with electrical constants σ, ϵ, and μ. The field component H_x (and H_y) in the ground can be set up as a superposition of elementary waves so that

$$H_x(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(K_1, K_2) e^{-i(K_1 x + K_2 y + K_3 z)} (K_1/K) dK_1 dK_2$$

where $f(K_1, K_2)$ is an aperature distribution and

$$K_1{}^2 + K_2{}^2 + K_3{}^2 = \omega^2 \mu \epsilon - i \sigma \mu \omega = K^2.$$

The electric field E_y is then given by

$$E_y = \frac{1}{\sigma + i\omega\epsilon} \frac{\partial H_x}{\partial z} \bigg]_{z=0} \text{ in the plane } z = 0.$$

It is noted that under the integral sign

$$\frac{\partial}{\partial z} = -iK_3 = -i\sqrt{K^2 - (K_1{}^2 + K_2{}^2)}$$

which can be expanded in a Taylor series as

$$\frac{\partial}{\partial z} = -iK \left[1 - \frac{K_1{}^2 + K_2{}^2}{2K^2} + O\left(\frac{1}{K^4}\right) \right].$$

The electric field tangential to the ground can then be written

$$E_y(x, y, 0) = -\eta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[1 - \frac{1}{2K^2}(K_1{}^2 + K_2{}^2) + \cdots \right] f(K_1, K_2) e^{-i(K_1 x + K_2 y)} (K_1/K) dK_1 dK_2$$

which is equivalent to

$$-E_y = \eta H_x + \frac{\eta}{2K^2} \left(\frac{\partial^2 H_x}{\partial x^2} - \frac{\partial^2 H_y}{\partial y^2} + 2 \frac{\partial^2 H_y}{\partial x \partial y} \right) + O\left(\frac{1}{K^4}\right).$$

A similar equation relates E_x and H_y in the plane $z=0$.

A more rigorous derivation of these relations[6] will appear in the *Journal of Applied Physics* in a forthcoming paper by W. J. Surtees and myself. I am also sending a copy of this letter to Dr. Dobrin.

Yours very truly,

JAMES R. WAIT

REFERENCES CITED IN DR. WAIT'S LETTER OF OCTOBER 29

1. H. G. Booker and P. C. Clemmow, Jour. Inst. Elect. Engs. Pt. III, *94*, 171, 1947 and Pt. III *97*, 11, 1950.
2. D. La Cour, Terr. Mag. and Atmos. Elect., *47*, 265, 1942.
3. L. Harang, Terr. Mag. and Atmos. Elect., *37*, 57, 1932.
4. B. Rolf, Terr. Mag. and Atmos. Elect., *36*, 7, 1931.
5. B. Angenheister, Terr. Mag. and Atmos. Elect., *25*, 26, 1920.
6. W. J. Surtees, Defence Research Board Report, University of Toronto, October 1952. (D. R. B. Grant No. 67) and J. R. Wait, Radio Physics Lab. Project Report 19-0-4, Sept., 1953.

Paris
16 November, 1953

Dr. James R. Wait
Defence Research Telecommunications Establishment
Ottawa, Ontario

Dear Sir:

Thank you for your last letter of October 29, 1953 and for the interesting proof which you have given me. I will make a brief reference to your work in the chapter "Electricité Tellurique" which Prof. Bartels of Gottingen has requested me to write for the next edition of *Handbuch der Physik.*

I am entirely in accord with you except regarding the question of the micropulsations being limited to a very restricted region. Only yesterday, M. Migaux, director of the Compagnie Générale de Geophysique, the man who "sees" the most micropulsations, told me that they do not exist except where there are industrial disturbances. Along with St. Thomas, we have to see to believe.

Please accept, dear Dr. Wait, this expression of my best wishes.

L. CAGNIARD
397, Rue de Vaugirard
Paris 15, France