*Zeitschrift für Geophysik, 1972, Band 38, Seite 257—289. Physica-Verlag, Würzburg*

# The Inverse Problem of Geomagnetic Induction

P. WEIDELT, Göttingen[1])

Eingegangen am 24. März 1972

*Summary:* The problem of revealing the electrical conductivity profile of a layered earth from geomagnetic induction data is solved using a modified version of the method of GEL'FAND and LEVITAN, originally devised for the solution of inverse Sturm-Liouville problems. The inversion procedure is applied to empirical data, which previously have been interpreted by a different method. – Since extensive use is made of the analytic properties of the response function in the complex frequency plane, these properties and related features of the response function are discussed at some length in an introductory section. Further it is shown that the inverse problem for a spherical earth can be transformed to the simpler problem for a flat earth and a uniform inducing field.

*Zusammenfassung:* Für die Umkehraufgabe der erdmagnetischen Tiefensondierung für horizontal geschichtete Leiter wird eine exakte Lösung angegeben. Es handelt sich dabei um eine modifizierte Fassung der Methode von GEL'FAND und LEVITAN zur Umkehrung Sturm-Liouvillescher Eigenwertaufgaben. Als Anwendungsbeispiel wird das Umkehrverfahren auf experimentelle Daten angewendet, die zuvor bereits nach einer anderen Methode interpretiert worden waren. – Das Umkehrverfahren macht wesentlich von den analytischen Eigenschaften der Beobachtungsdaten in der komplexen Frequenzebene Gebrauch. Deshalb werden in einem einleitenden Abschnitt ausführlich diese Eigenschaften und ihre Konsequenzen behandelt. Ferner wird gezeigt, daß sich die Umkehraufgabe für eine kugelförmige Erde auf den einfacheren Fall einer ebenen Erde mit einem homogenen induzierenden Feld reduzieren läßt.

## 1. Introduction

Geomagnetic induction data are generally interpreted by assuming an electrical conductivity model with several free parameters, which in turn are adjusted to the data either by curve fitting or by analytic methods. However, if the conductivity changes with depth only, direct inversion without recourse to model calculations becomes possible. This inverse problem has been solved first by SIEBERT [1964], and in a slightly modified version by ČETAEV [1966], both using the WBK-approximation. The shortcoming of their method is that the whole conductivity profile is recovered only from the asymptotic behaviour of the response function for high frequencies. Therefore highly precise data are required in this frequency range, whereas the valuable information of the low frequency part is not exhausted. However, by giving an algo-

[1]) Dr. PETER WEIDELT, Institut für Geophysik der Universität Göttingen, 34 Göttingen, Herzberger Landstraße 180.

rithm it was implicitly shown that the conductivity distribution can be inferred uniquely from the response function. The question of uniqueness has been treated explicitly by TICHONOV [1965], and more recently by BAILEY [1970], who formulated an integral constraint in the frequency domain, from which the conductivity profile can be deduced uniquely.

The present paper is concerned with an alternative solution of the inverse problem, which is essentially a modified version of the method of GEL'FAND & LEVITAN [1951 a, b] for the solution of the inverse Sturm-Liouville problem. The Gel'fand-Levitan procedure has found much attention in connection with the inverse problem in quantum-mechanical scattering theory (cf. the review article of FADDEEV [1963]), and has quite recently been applied by JOHNSON & SMYLIE [1971] to reveal the conductivity distribution in the lower mantle, assuming a knowledge of the time constants, which govern the diffusion of magnetic fields from the core-mantle boundary upwards. Although JOHNSON & SMYLIE and the present author refer to the same sources, their approaches differ significantly both in the information assumed and in the method.

At a first glance a direct inversion procedure appears to be very attractive, since it is less biased by preconceived models than parameter adjustment techniques. In practice, however, it loses much of its appeal by the fact that the inverse problem of geomagnetic induction belongs to the large class of improperly posed problems [STRACHOV 1969, ANDERSSEN 1970], where small changes in the data can cause large changes in the results. Due to the inherent scatter of the data a result obtained by direct inversion represents just one element of the set of feasible solutions, and cannot deserve more attention than any other feasible solution obtained by different means. Often approximate methods are fully adequate to the quality of the data. A quite simple but powerful approximate solution of this kind has been proposed by SCHMUCKER [1970, p. 69].

Despite the proviso mentioned above, a treatment of the inverse problem of geomagnetic induction appears to be justified by the fact that it is one of the rare geophysical inverse problems, which allows an exact solution. Moreover, in the course of this study some general properties of the response function can be derived, which are of interest for any inversion procedure. Since these results are not well known (although the underlying theory is essentially the theory of ordinary linear second order differential equations), Secs. 2 and 3 contain a detailed investigation of these properties. The inversion procedure itself is described in Secs. 4 and 5, and is illustrated by examples in Secs. 6 and 7.

## 2. Properties of the response function

For simplicity, only a flat earth and a uniform inducing magnetic field are considered here. Effects of a non-uniform magnetic field and the curvature of the earth are afterwards taken into account by simple transformations (cf. Sec. 3). As a basic limitation the (isotropic) electrical conductivity $\sigma$ is assumed to vary with depth $z$ only

($z$ positive downwards). Neglecting the displacement current, assuming vacuum permeability and a harmonic time factor $e^{+i\omega t}$ throughout, the complex amplitudes $E(z, \omega)$ and $H(z, \omega)$ of the horizontal electric and magnetic field (in $y$ and $x$ direction, respectively) are interconnected by

$$H'(z,\omega)=\sigma(z)\ E(z,\omega), \tag{2.1}$$

$$E'(z,\omega)=i\omega\mu_0\ H(z,\omega), \tag{2.2}$$

SI-units being used. The prime always denotes differentiation with respect to the (first) argument. Elimination of $H$ leads to

$$E''(z,\omega)=i\omega\mu_0\sigma(z)E(z,\omega). \tag{2.3}$$

The response function $c(\omega)$ is defined as

$$c(\omega)=-\frac{E(0,\omega)}{E'(0,\omega)}=-\frac{E(0,\omega)}{i\omega\mu_0\,H(0,\omega)}. \tag{2.4}$$

Its relation to the apparent resistivity $\varrho_a$ of magnetotellurics [CAGNIARD 1953] is

$$\varrho_a(\omega)=\omega\mu_0\,|c(\omega)|^2. \tag{2.5}$$

Let $z_m$ be the greatest depth to which the electromagnetic field can penetrate, i. e.

$$z_m=\begin{cases}\infty, \text{ if there is no perfect conductor}\\ \text{else the depth of the perfect conductor}\end{cases}. \tag{2.6}$$

Then the problem to be solved may be stated as follows:

Given $c(\omega)$ in $0 < \omega < \infty$, wanted $\sigma(z)$ in $0 \leq z < z_m$.

In principle the necessary information can be reduced, since $c(\omega)$ turns out to be an analytic function, which is completely specified by its values in an arbitrary small interval.

Some properties of the response function $c(\omega)$ are now listed for later reference.

*a) Analytic properties in the complex frequency plane*

The response function $c(\omega)$ is zero-free and analytic in the whole $\omega$-plane except on the positive imaginary axis. Here it has either an infinite series of interlacing simple poles and zeros, or a finite number (which may be null) of poles and zeros and two branch points (one at $\omega = +i\infty$), according whether the integral

$$\lim_{z \to z_m} \int_0^z \sqrt{\sigma(t)}\,\mathrm{d}t \tag{2.7}$$

converges or not. The same applies to the normalized electric field $E(z, \omega)/E(0, \omega)$. A possible perfect conductor at $z = z_m$ is not to be included in (2.7).

The proofs follow from general theorems on second order linear differential equations (e.g. TITCHMARSH 1962) and are only indicated here. Let $w_1(z, \omega)$ and $w_2(z, \omega)$ be two solutions of (2.3) with the initial conditions

$$w_1(0,\omega) = 1\,, \quad w_1'(0,\omega) = 0\,, \quad w_2(0,\omega) = 0\,, \quad w_2'(0,\omega) = 1\,. \tag{2.8}$$

Since their Wronskian

$$w_1(z)\,w_2'(z) - w_2(z)\,w_1'(z) = 1 \tag{2.9}$$

does not vanish, the solutions are linearly independent for all $z$, and the actual solution $E$ is a linear combination of them:

$$E(z,\omega)/E(0,\omega) = w_1(z,\omega) - w_2(z,\omega)/c(\omega)\,. \tag{2.10}$$

Away from the positive imaginary $\omega$-axis $E(z_m, \omega)$ is a constant, which differs from zero only, if $\sigma(z)$ decreases for $z \to \infty$ faster than $z^{-2}$. Since in this case $w_1(z, \omega)$ tends to infinity, Eq. (2.10) yields for any conductivity profile

$$c(\omega) = \lim_{z \to z_m} \frac{w_2(z,\omega)}{w_1(z,\omega)}\,. \tag{2.11}$$

The nature of the singularities of $c(\omega)$ can be investigated as follows. The solutions $w_1$ and $w_2$ are entire functions of $\omega$, i.e. they are free of singularities in the finite $\omega$-plane (e.g. [TITCHMARCH 1962], p. 6). Multiply the differential equation (2.3) for $E = w_m$, $m = 1, 2$, by the complex-conjugate solution $w^*_m$, integrate over $z$, and obtain after integration by parts (on using (2.8))

$$w_m^*(z)\,w_m'(z) = \int_0^z \{|w_m'(t)|^2 + i\omega\mu_0\sigma(t)\,|w_m(t)|^2\}\,\mathrm{d}t\,, \quad m = 1, 2\,. \tag{2.12}$$

Hence, all zeros of $w_1$ and $w_2$ lie on the positive imaginary $\omega$-axis, where they constitute the poles and zeros of the meromorphic function

$$c(z,\omega) = w_2(z,\omega)/w_1(z,\omega)\,, \tag{2.13}$$

which is the response function for the case that the conductivity at depths greater than $z$ is replaced by a perfect conductor at depth $z$. On the positive imaginary axis put $\omega = i\lambda$, $\lambda > 0$. Denote the $n$-th zero of $w_m(z, i\lambda)$ by $\lambda_{mn}$ and $\partial w_m/\partial\lambda$ by $\dot{w}_m$. Then multiply on one hand (2.3) for $E = w_m$ by $\dot{w}_m$, differentiate on the other hand (2.3) with respect to $\lambda$ and multiply by $w_m$, integrate the difference over $z$, and obtain after integration by parts (on using (2.8) and the fact that $w_m(z, i\lambda_{mn})$ is real)

$$\dot{w}_m(z, i\lambda_{mn}) \cdot w_m'(z, i\lambda_{mn}) = \int_0^z \mu_0 \sigma(t)\, w_m^2(t, i\lambda_{mn})\, \mathrm{d}t > 0\,,$$

or in virtue of (2.9)

$$\dot{w}_1(z, i\lambda_{1n})/w_2(z, i\lambda_{1n}) < 0\,, \quad \dot{w}_2(z, i\lambda_{2n})/w_1(z, i\lambda_{2n}) > 0\,. \tag{2.14}$$

Since $\dot{w}_m(z, i\lambda_{mn})$ does not vanish, the zeros are simple. Further it is easily deduced from (2.14) that between two successive zeros of $w_1$ there must be an odd number of zeros of $w_2$, and vice versa. Hence, the zeros interlace.

The distance $\Delta\lambda_n$ between two successive zeros of $w_1$ or $w_2$ is for large $n$ asymptotically given by

$$\Delta\lambda_n = 2n\,[\pi/J(z)]^2\,, \quad J(z) = \int_0^z \sqrt{\mu_0 \sigma(t)}\, \mathrm{d}t$$

(e.g. MORSE & FESHBACH 1953, p. 739). Therefore, the density of poles and zeros increases when $z$ is enhanced, and the analytic behaviour of

$$c(\omega) = \lim_{z \to z_m} c(z, \omega) \tag{2.15}$$

depends on the behaviour of $J(z)$ for $z \to z_m$. If $J(z)$ remains finite, there is an infinite series of poles and zeros; if $J(z)$ diverges the isolated poles and zeros beyond a certain limit point merge into a branch cut from that point to $\omega = +i\infty$, whereas below the lower branch point a finite number of poles and zeros may subsist. – The analytic properties of $E(z, \omega)/E(0, \omega)$ follow from the properties of $c(\omega)$ and (2.10).

Poles and branch cut of $c(\omega)$ define the discrete and continuous spectrum of decay constants of freely decaying horizontally uniform current systems within the conductor. This is a consequence of (2.4) and the fact that the associated magnetic field, which cannot be observed outside the conductor [PRICE 1950], has to vanish at $z = 0$.

Two examples will illustrate the preceding results. First consider the uniform half-space with $\sigma(z) = \sigma_0$. Let

$$k = \sqrt{i\omega\mu_0\sigma_0}\,.$$

Then $w_1 = \cosh kz$, $w_2 = k^{-1} \sinh kz$, both being entire functions of $\omega$, since their power series representations contain only even powers of $k$. The poles and zeros of $c(z, \omega) = k^{-1} \tanh kz$ lie at

$$\omega_{1n} = \frac{i}{\mu_0 \sigma_0} \left( \frac{2n-1}{2} \frac{\pi}{z} \right)^2 \quad \text{and} \quad \omega_{2n} = \frac{i}{\mu_0 \sigma_0} \left( n \frac{\pi}{z} \right)^2, \quad n = 1, 2, \ldots$$

For $z \to \infty$ they cluster at $\omega = +i0$, which gets a branch point of $c = k^{-1}$. The other branch point is $\omega = i\infty$.

Next consider the conductivity profile

$$\sigma(z) = \sigma_0 \{1 - 2bz + (b^2 - a^2) z^2\}^{-2}, \quad a > 0 \tag{2.16}$$

treated by WEIDELT [1970, p. 30]. For $b \geq 0$ there is a monotone increase of $\sigma$, getting infinite at $z_m = 1/(a+b)$, and

$$c(\omega) = (b + \sqrt{a^2 + k^2})^{-1}, \quad k = \sqrt{i\omega\mu_0\sigma_0}. \tag{2.17}$$

The singularities of $c$ are two branch points at $\omega = ia^2/\mu_0\sigma_0$ and $\omega = i\infty$. For $-a < b < 0$ the conductivity first decreases to a minimum, and then increases to infinity. Again $c$ is given by (2.17), but now an additional pole at $\omega = i(a^2 - b^2)/\mu_0\sigma_0$ occurs. Finally let $b < -a$. Then there is a monotone decrease of conductivity, $J(z)$ remains finite for $z \to \infty$, and

$$c(\omega) = [b - \sqrt{a^2 + k^2} \coth \{\sqrt{1 + k^2/a^2} \operatorname{arccoth}(b/a)\}]^{-1}$$

has an infinite series of poles and zeros (but no branch points!).

In Sec. 5 and Appendix A a representation of the response function in terms of its singularities is required. If $J(z_m)$ is finite, $c(\omega)$ is a meromorphic function with simple poles at $\omega = i\lambda_{1n}$ and permits by the Mittag-Leffler theorem (e.g. MORSE & FESHBACH 1953, p. 383) an expansion in partial fractions:

$$c(\omega) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_{1n} + i\omega}, \quad a_n > 0, \quad \lambda_{1n} > 0. \tag{2.18}$$

This representation is justified due to

$$a_n = \lim_{z \to z_m} \lim_{\omega \to i\lambda_{1n}} (\lambda_{1n} + i\omega) c(z, \omega) = -w_2(z_m, i\lambda_{1n}) / \dot{w}_1(z_m, i\lambda_{1n}) > 0$$

(on using (2.14) and (2.15)), and $\lambda_{1n} = O(n^2)$, $a_n = O(1)$ for $n \to \infty$ (cf. MORSE & FESHBACH 1953, p. 739). In the general case (2.18) must be replaced by

$$c(\omega) = \int_0^\infty \frac{a(\lambda)\,d\lambda}{\lambda + i\omega}, \quad a(\lambda) \geq 0, \tag{2.19}$$

where $a(\lambda)$ is a generalized function to include both the discrete and the continuous part of the spectrum. (Alternatively Stieltjes integral notation would be appropriate.) The non-decreasing function $\int a(\lambda)\,d\lambda$ is known as the *spectral function.*

*b) Symmetry relation for $c(\omega)$*

$c(\omega)$ satisfies

$$c(-\omega^*) = c^*(\omega), \tag{2.20}$$

i.e. it takes conjugate values at two points symmetric to the axis of imaginaries. Eq. (2.20) follows with (2.11) from the fact that $w^*_m(z, \omega)$ and $w_m(z, -\omega^*)$, $m=1, 2$, satisfy the same differential equations and initial conditions. Hence, they are identical.

*c) Limiting values for large and small frequencies*

For large frequencies

$$c(\omega) = k^{-1} - \frac{1}{4}\sigma'(0)/\sigma(0)\,k^{-2} + O(k^{-3}), \quad \omega \to \infty, \quad k^2 = i\omega\mu_0\sigma(0), \tag{2.21}$$

which may be obtained by a WBK-approximation (e.g. KAMKE 1959, p. 138, SIEBERT 1964), and for small frequencies

$$\lim_{\omega \to 0} c(\omega) = z_m, \tag{2.22}$$

following from (2.11) with $w_1 = 1$, $w_2 = z$.

*d) Dispersion relations*

Because of the analytic properties of $c(\omega)$, its real and imaginary part are not independent functions of frequency. Let $\omega_0$ be a point in the upper $\omega$-plane and $C$ be a closed contour consisting of the real axis and a large semicircle in the lower half-plane. Then

$$\frac{1}{\pi i}\int_C \frac{c(\omega')\,d\omega'}{\omega' - \omega_0} = 0,$$

since the integrand is analytic in $C$. Due to (2.21) the large semicircle does not contribute, and the contour can be confined to the real axis. Here put $\omega' = x$ and let $\omega_0 = \omega + i\varepsilon$ ($\omega$ real, $\varepsilon > 0$) tend to the real axis. Then

$$0 = \lim_{\varepsilon \to +0} \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{c(x)\,\mathrm{d}x}{x-\omega-i\varepsilon} = c(\omega) + \frac{1}{\pi i} \fint_{-\infty}^{+\infty} \frac{c(x)\,\mathrm{d}x}{x-\omega}, \tag{2.23}$$

where

$$\lim_{\varepsilon \to +0} \frac{1}{\pi} \frac{\varepsilon}{(x-\omega)^2+\varepsilon^2} = \delta(x-\omega)$$

has been used. $\fint$ denotes the Cauchy principal value. Let for real frequencies

$$c(\omega) = g(\omega) - ih(\omega), \tag{2.24}$$

where in virtue of (2.20) $g(-\omega) = g(\omega)$, $h(-\omega) = -h(\omega)$. Hence, a separation of (2.23) in its real and imaginary part yields

$$g(\omega) = \frac{1}{\pi} \fint_{-\infty}^{+\infty} \frac{h(x)\,\mathrm{d}x}{x-\omega} = \frac{2}{\pi} \fint_0^{\infty} \frac{xh(x)\,\mathrm{d}x}{x^2-\omega^2}, \tag{2.25 a}$$

$$h(\omega) = -\frac{1}{\pi} \fint_{-\infty}^{+\infty} \frac{g(x)\,\mathrm{d}x}{x-\omega} = -\frac{2}{\pi} \fint_0^{\infty} \frac{\omega g(x)\,\mathrm{d}x}{x^2-\omega^2}. \tag{2.25 b}$$

Relations of this kind, occurring in many branches of physics, are well known as *dispersion relations*. They are a consequence of the causality requirement (e.g. LANDAU & LIFSCHITZ 1966/67, v. 3 § 129, v. 5 § 125, v. 8 § 62 and 67; BAILEY 1970; WEIDELT 1970, p. 23). Relations corresponding to (2.25a, b) also exist for modulus and phase of $c(\omega)$. Since $c$ is free of zeros in the lower half-plane, the function

$$\log\{\sqrt{i\omega\mu_0\sigma(0)}\,c(\omega)\}$$

is analytic there and vanishes for $|\omega| \to \infty$ due to (2.21). Put

$$c(\omega) = |c(\omega)|\,e^{-i\psi(\omega)} \tag{2.26}$$

and assume $\omega > 0$. Then the relation corresponding to (2.25b) is

$$\psi(\omega) = \frac{\pi}{4} - \frac{2\omega}{\pi} \fint_0^{\infty} \log\{\sqrt{x\mu_0\sigma(0)}\,|c(x)|\} \frac{\mathrm{d}x}{x^2-\omega^2},$$

or introducing the apparent resistivity $\varrho_a(\omega)$ by (2.5):

$$\psi(\omega)=\frac{\pi}{4}-\frac{\omega}{\pi}\int_0^\infty \log\{\varrho_a(x)/\varrho_0\}\frac{dx}{x^2-\omega^2}, \tag{2.27}$$

where $\varrho_0=1/\sigma(0)$. By (2.27) the phase of experimental data can be deduced from the apparent resistivity, which is often better accessible. There exists a simple approximate version of (2.27). Integration by parts yields

$$\psi(\omega)-\frac{\pi}{4}=\frac{1}{2\pi}\int_0^\infty \frac{d\log\varrho_a(x)}{dx}\cdot\log\left|\frac{\omega-x}{\omega+x}\right|dx=$$

$$=\frac{1}{2\pi}\int_0^\infty \frac{d\log\varrho_a(x)}{d\log x}\cdot\log\left|\frac{\omega-x}{\omega+x}\right|\frac{dx}{x},$$

or since $x^{-1}\log|(\omega-x)/(\omega+x)|$ almost behaves like a $\delta$-function

$$\psi(T)\approx\frac{\pi}{4}\{1+d\log\varrho_a(T)/d\log T\}, \tag{2.28}$$

where $T$ is the period, and the result

$$\int_0^\infty \log\left|\frac{\omega-x}{\omega+x}\right|\frac{dx}{x}=2\int_0^1\log\left(\frac{1-t}{1+t}\right)\frac{dt}{t}=-\frac{\pi^2}{2}$$

has been applied. Since double-logarithmic plots of $\varrho_a(T)$ are used, a first approximation of the phase can immediately be obtained from the slope of the sounding curve. Fig. 1 gives two examples.

It should be noted that relations corresponding to (2.25a, b) exist for all realizable linear systems, whereas relations between modulus and phase can be given only for the restricted class of transfer functions, which are free of zeros in the lower frequency plane (minimal phase systems).

*e) Inequalities*

Let $\omega>0$ and define an operator D by

$$\mathrm{D}f=\omega\, df/d\omega=df/d\log\omega=-df/d\log T. \tag{2.29}$$

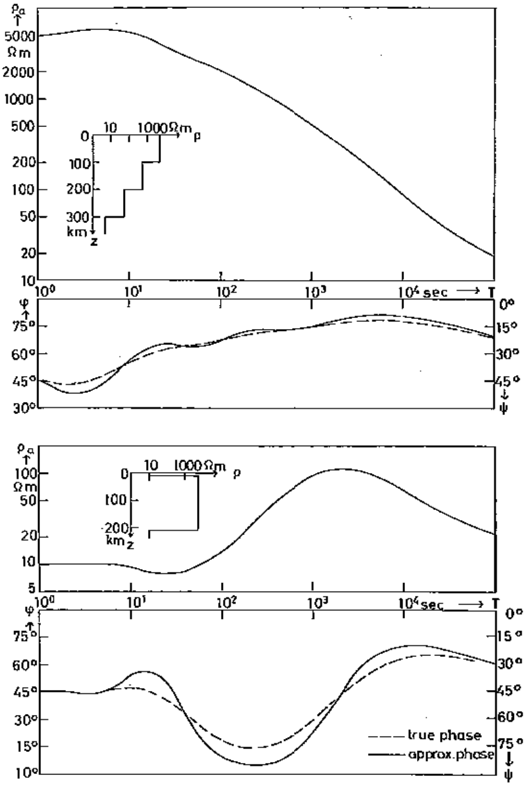



Fig. 1: Two examples for the determination of the phase from the $\varrho_a$-curve using the approximation (2.28). The angle $\varphi = 90° - \psi$ is the phase angle between electric and magnetic field.

Then (recalling the definition (2.24)) the following inequalities apply:

$$g \geq 0, \quad h \geq 0, \tag{2.30 a, b}$$

$$\mathrm{D}g \leq 0, \tag{2.31}$$

$$0 \leq -\mathrm{D}\,|c| \leq |c|, \tag{2.32 a, b}$$

$$|\mathrm{D}c| \leq h, \quad |c + \mathrm{D}c| \leq g, \tag{2.33 a, b}$$

$$|\mathrm{D}^2 c| \leq h, \quad |c + 2\,\mathrm{D}c + \mathrm{D}^2 c| \leq g. \tag{2.34 a, b}$$

Alternatively these constraints can be expressed in terms of apparent resistivity $\varrho_a$ (cf. (2.5)) and phase $\psi$ (cf. (2.26)). For example (2.30a, b), (2.32a, b), (2.33a, b) then read:

$$0 \leq \psi \leq \pi/2, \tag{2.30 a, b)'}$$

$$-\varrho_a \leq \mathrm{D}\varrho_a \leq \varrho_a, \tag{2.32 a, b)'}$$

$$\frac{1}{4}(1-\mathrm{D}\varrho_a/\varrho_a)^2+(\mathrm{D}\psi)^2 \leq \sin^2\psi, \tag{2.33 a)'}$$

$$\frac{1}{4}(1+\mathrm{D}\varrho_a/\varrho_a)^2+(\mathrm{D}\psi)^2 \leq \cos^2\psi. \tag{2.33 b)'}$$

Hence, (2.33) implies (2.32). (The quantity $-\mathrm{D}\varrho_a/\varrho_a$ is the slope of the sounding curve $\varrho_a(T)$ in a double-logarithmic plot.)

The proofs of (2.30)–(2.34) follow almost immediately from the representation (2.19). Together with additional constraints they are given in Appendix A. If experimental data do not fit these inequalities, some of the underlying assumptions on conductivity and external field are definitely wrong. In this case we are able to compute a set of "corrected" data, which satisfy the inequalities thereby deviating least (in a given norm) from the original data. This leads to a problem in convex programming, which is easily solved by the cutting-plane method (e.g. Collatz & Wetterling 1971, p. 124). An example is given in Fig. 2. The original data, response values for the first four Sq-harmonics, were obtained by Schmucker [1971, private communication] as an average for South East Europe. – Derivatives were determined from the

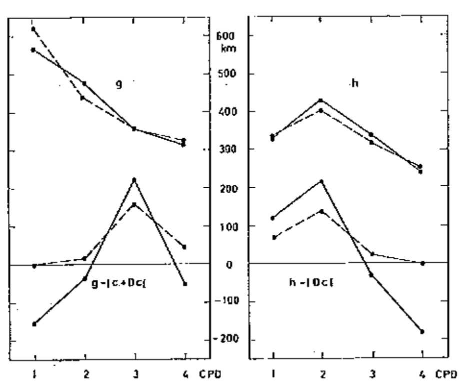



Fig. 2: An example for the optimal correction of experimental data, which do not satisfy the constraints (2.33a, b) everywhere. Input data and corrected data are connected by full and dashed lines, respectively. CPD means "cycles per day".

slope of a parabola through three successive points, and the least squares norm for the relative deviations has been used. Any interpretation of the original data g and h can give no better fit than that indicated by the broken lines. The application of the above constraints, which were obtained for a flat earth and a uniform external field, to problems with spherical symmetry is justified due to the results of Sec. 3.

*f) Computation of the response function*

When the conductivity is recovered by any inversion scheme, the response function $c(\omega)$ has to be computed for check with the input data. This can be done by (2.11), where $w_1$ and $w_2$ are obtained by numerical integration of (2.3) with the initial values from (2.8). The integration has to proceed downward until the ratio $w_2/w_1$ tends to a limit. For real frequencies the moduli of $w_1$ and $w_2$ steadily increase with depth, since

$$\frac{\mathrm{d}}{\mathrm{d}z}|w_m(z,\omega)|^2 = 2\,Re\,\{w_m^*(z,\omega)\,w_m'(z,\omega)\}\,, \quad m=1,2$$

is positive in virtue of (2.12). Let $f = w_2/w_1$. Then $f' = w_1^{-2}$, using (2.9). Hence, (2.11) can be replaced by

$$c(\omega) = \int_0^{z_m} \frac{\mathrm{d}z}{w_1^2(z,\omega)}. \tag{2.35}$$

Thus only $w_1$ is required. An alternative method has been proposed by ECKARDT [1968], who reduced (2.3) to a Riccati equation, which was solved by upward integration with an arbitrary initial value at a sufficiently deep starting point. The fastest method, however, is the approximation of the conductivity profile by a set of homogeneous layers, for which $c(\omega)$ can be computed by well-known recurrence formulae.

*g) Physical meaning of the real part of $c(\omega)$*

The real part $g(\omega)$ of the response function admits a simple physical interpretation. Let

$$j(z,\omega) = \sigma(z)\,E(z,\omega) \tag{2.36}$$

be the density of the induced currents. Then

$$\int_0^{z_m} j(z,\omega)\,\mathrm{d}z = H(0,\omega)\,, \quad \int_0^{z_m} z\,j(z,\omega)\,\mathrm{d}z = -\frac{1}{i\omega\mu_0}\,E(0,\omega)\,,$$

which is easily verified by partial integration on using (2.36), (2.1), and (2.2). Hence, taking the phase of $H(0,\omega)$ as reference phase and applying (2.4),

$$g(\omega) = \int_0^{z_m} z\,Re\,\{j(z,\omega)\}\,\mathrm{d}z \Big/ \int_0^{z_m} Re\,\{j(z,\omega)\}\,\mathrm{d}z\,.$$

Thus, in a mechanical analogy, the positive length $g(\omega)$ can be interpreted as the depth of the "centre of gravity" of the in-phase induced current system. In accordance with well established ideas regarding the induction process, $g'(\omega) < 0$ (Eq. 2.31) shows that the mean depth of the current system increases if the frequency is diminished. Limiting values are $g(\infty)=0$ and $g(0)=z_m$ (cf. (2.21) and (2.22)). The present interpretation of $g(\omega)$ is basic for the inversion procedure of SCHMUCKER [1970, p. 69].

## 3. Arbitrary external field and spherical earth

So far only a uniform inducing field and a flat earth have been considered. Retaining the assumption that the conductivity shall vary with depth only, the electric field vector $\boldsymbol{E}$ remains a tangential solenoidal vector for any solenoidal inducing field and for both a flat and a spherical earth [LAHIRI & PRICE 1939, PRICE 1950, YUKUTAKE 1967, ECKARDT 1968]. $\boldsymbol{E}$ satisfies

$$\Delta \boldsymbol{E}(\boldsymbol{r}, t) = \mu_0 \sigma \frac{\partial}{\partial t} \boldsymbol{E}(\boldsymbol{r}, t), \tag{3.1}$$

where $\boldsymbol{r}$ is the vector of position. Its representation as a superposition of the particular solutions of (3.1) is

a) for a flat earth:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \iiint_{-\infty}^{+\infty} a(\boldsymbol{\varkappa}, \omega)\, w(z, \varkappa, \omega)\, \hat{\boldsymbol{z}} \times \operatorname{grad}\{e^{i(\boldsymbol{\varkappa}\cdot\boldsymbol{r}+\omega t)}\}\, d\varkappa_x\, d\varkappa_y\, d\omega, \tag{3.2 a}$$

b) for a spherical earth:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \int_{-\infty}^{+\infty} d\omega \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} a_n^m(\omega)\, w_n(r, \omega)\, \hat{\boldsymbol{r}} \times \operatorname{grad}\{P_n^m(\cos\theta)\, e^{i(m\phi+\omega t)}\}. \tag{3.2 b}$$

Here $\boldsymbol{\varkappa} = \varkappa_x \hat{\boldsymbol{x}} + \varkappa_y \hat{\boldsymbol{y}}$ is the horizontal wave vector, $\varkappa = \sqrt{\boldsymbol{\varkappa}\cdot\boldsymbol{\varkappa}}$ the wave number, $\theta$ the colatitude, $\varphi$ the longitude, $P^m{}_n$ the associated Legendre function, and $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{r}}$ unit vectors in the direction of increasing $x$, $y$, $z$, $r$. The functions $a(\boldsymbol{\varkappa}, \omega)$ and $a^m{}_n(\omega)$ represent the spectral density of the inducing field in the space and time domain. The response of the conductor to the corresponding harmonics is described by $w(z, \omega, \varkappa)$ and $w_n(r, \omega)$, which satisfy

$$w''(z, \omega, \varkappa) = \{\varkappa^2 + i\omega\mu_0\sigma(z)\}\, w(z, \omega, \varkappa) \tag{3.3 a}$$

and

$$w_n''(r, \omega) = \left\{\frac{n(n+1)}{r^2} + i\omega\mu_0\sigma(r)\right\} w_n(r, \omega). \tag{3.3 b}$$

The response functions

$$c(\omega) = -\frac{w(0, \omega, \varkappa)}{w'(0, \omega, \varkappa)} \quad \text{and} \quad c(\omega) = +\frac{w_n(R, \omega)}{w_n'(R, \omega)} \tag{3.4a, b}$$

($R$ being the radius of the earth) can be obtained after harmonic analysis – both in space and time – of the electromagnetic field at the surface of the earth on three different ways:

1. from the ratio of orthogonal tangential electric and magnetic field components,
2. from the ratio of normal and tangential magnetic field components,
3. from the ratio of internal and external parts of the magnetic field.

Moreover, there is the possibility to determine the response function from the ratio of the vertical gradient of a horizontal magnetic field component just beneath the surface to the component at the surface [MEYER 1966].

The inverse problem is reduced to the inverse problem for a flat earth and a uniform external field by the transformations

$$\left.\begin{aligned} &\tilde{z} = \varkappa^{-1} \tanh \varkappa z \\ &\tilde{w}(\tilde{z}, \omega) = w(z, \omega)/\cosh \varkappa z \\ &\tilde{\sigma}(\tilde{z}) = \sigma(z) \cdot \cosh^4(\varkappa z) \end{aligned}\right| \quad \begin{aligned} &\tilde{z} = R\frac{\varrho^{-n} - \varrho^{n+1}}{(2n+1) f(\varrho)}, \\ &\tilde{w}(\tilde{z}, \omega) = w_n(r, \omega)/f(\varrho), \\ &\tilde{\sigma}(\tilde{z}) = \sigma(r) \cdot f^4(\varrho), \end{aligned} \tag{3.5a, b}$$

where $\varrho = r/R$ and $f(\varrho) = \{(n+1)\,\varrho^{-n} + n\varrho^{n+1}\}/(2n+1)$. They transform (3.2a, b) into

$$\tilde{w}''(\tilde{z}, \omega) = i\omega\mu_0 \tilde{\sigma}(\tilde{z})\, \tilde{w}(\tilde{z}, \omega),$$

which is (2.3) for a flat earth and a uniform external field, and do not affect the response functions, i.e.

$$-\tilde{w}(0, \omega)/\tilde{w}'(0, \omega) = -w(0, \omega)/w'(0, \omega) = w_n(R, \omega)/w_n'(R, \omega).$$

Hence, any $c(\omega)$ produced by an external field with wave number $\varkappa$ or spherical harmonic of degree $n$ (where $\varkappa$ and $n$ are assumed to be independent of $\omega$) can first be interpreted by a uniform field and a flat earth, and the resulting profile $\tilde{\sigma}(\tilde{z})$ is then transformed into the true distribution by

$$\sigma(z) = \cosh^{-4} \varkappa z \cdot \tilde{\sigma}(\varkappa^{-1} \tanh \varkappa z), \tag{3.6a}$$

$$\sigma(r) = f^{-4}(\varrho) \cdot \tilde{\sigma}\left(R\frac{\varrho^{-n} - \varrho^{n+1}}{(2n+1) f(\varrho)}\right). \tag{3.6b}$$

The physical basis of the preceding transformations is the fact that damping by a *perfect* conductor and geometrical attenuation are equivalent, enter into the response function in the same way, and cannot be separated without additional information. Consider for illustration a perfect conductor at $z = z_m$ and an inducing field with wave number $\varkappa$. Then $c(\omega) = \varkappa^{-1} \tanh \varkappa z_m = c_0 = \text{const.}$, for the solutions $w_1$ and $w_2$ of (3.3a) are $\cosh \varkappa z$ and $\varkappa^{-1} \sinh \varkappa z$. Hence, given a response function $c(\omega) = c_0$, it can be interpreted by a perfect conductor at $z_m = \varkappa^{-1} \tanh^{-1} \varkappa c_0$ with $0 \leq \varkappa \leq 1/c_0$. In the limit $\varkappa = 0$, $c(\omega)$ is explained by a uniform external field and a perfect conductor at $z_m = c_0$, whereas in the limit $\varkappa = 1/c_0$ there is only geometrical attenuation and no perfect conductor.

Consequently, the profile $\tilde{\sigma}(\tilde{z})$ for a uniform field always includes a perfect conductor, which is lowered, when passing by (3.6a) to the profile $\sigma(z)$, thereby replacing electromagnetic damping fully or in part by geometrical damping. Interpretation of a response function $c(\omega)$ by external fields with wave number $\varkappa$ or degree $n$ is subject to the restriction

$$c(0) \leq \varkappa^{-1} \quad \text{or} \quad c(0) \leq R/(n+1), \tag{3.7}$$

a consequence of (3.4a, b) and the fact that the relevant solutions of (3.3a, b) in the limit $\omega \to 0$ are given by $e^{-\varkappa z}$ and $r^{n+1}$ (provided that there is no perfect conductor). — The transformations (3.5) are special cases of a more general class, which is given in Appendix B.

## 4. Solution of the inverse problem

In this section it is shown, how the conductivity profile $\sigma(z)$ can be deduced from the response function $c(\omega)$. The adopted procedure is essentially based on the method of Gel'fand & Levitan [1951 a, b] for the solution of the inverse Sturm-Liouville problem. The special needs of the inverse geomagnetic induction problem, however, introduce substantial modifications of the original approach.

Let $\sigma(z)$ have discontinuities only in its derivatives. Then by the substitutions

$$\omega \to k = \sqrt{i\omega\mu_0\sigma(0)}, \tag{4.1}$$

$$z \to x = \int_0^z \sqrt{\sigma(t)/\sigma(0)}\, \mathrm{d}t, \tag{4.2}$$

$$E(z, \omega) \to f(x, k) = \sqrt[4]{\sigma(z)/\sigma(0)}\, E(z, \omega)/E(0, \omega), \tag{4.3}$$

$$\sigma(z) \to u(x) = \sqrt[4]{\sigma(z)/\sigma(0)} \tag{4.4}$$

the differential equation (2.3) is transformed into

$$f''(x,k)=\{k^2+V(x)\}f(x,k), \tag{4.5}$$

where

$$V(x)=u''(x)/u(x) \tag{4.6}$$

The new variable $k$ has the dimension of a reciprocal length. Choose that branch of (4.1), which maps the upper sheet of the $\omega$-plane into the right half of the $k$-plane. Then the positive and negative $\omega$-axis is mapped into the bisectors of the first and fourth quadrant, respectively. Since $c(\omega)$ and $E(z,\omega)/E(0,\omega)$ are analytic outside the positive imaginary $\omega$-axis (cf. Sec. 2), the quantities

$$c(k)\equiv c(\omega) \tag{4.7}$$

and $f(x,k)$ are analytic to the right of the imaginary $k$-axis. With respect to (4.1) the symmetry relation (2.20) now reads

$$c(k^*)=c^*(k), \tag{4.8}$$

i.e. symmetry about the real $k$-axis.

Let $f_+(x,k)$ and $f_-(x,k)$ be two solutions of (4.5) with initial conditions

$$f_\pm(0,k)=1\,,\quad f'_\pm(0,k)=u'(0)\pm k\,. \tag{4.9}$$

These functions can be represented by

$$f_\pm(x,k)=e^{\pm kx}+\int_{-x}^{+x}A(x,t)\,e^{\pm kt}\,\mathrm{d}t\,, \tag{4.10}$$

where $A$ is real and independent of $k$. This representation is justified as follows: Insert $f_+$, say, into (4.5), integrate the term

$$k^2\int_{-x}^{+x}A(x,t)\,e^{kt}\,\mathrm{d}t=\int_{-x}^{+x}A(x,t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{kt}\,\mathrm{d}t\,,$$

occurring at the right-hand side of (4.5), two times by parts, and use the identity

$$\left.\frac{\partial A}{\partial x}\right|_{t=\pm x}\pm\left.\frac{\partial A}{\partial t}\right|_{t=\pm x}=\frac{\mathrm{d}}{\mathrm{d}x}A(x,\pm x)\,.$$

The final result is

$$e^{kx}\left\{2\frac{\mathrm{d}}{\mathrm{d}x}A(x,x)-V(x)\right\}+e^{-kx}\frac{\mathrm{d}}{\mathrm{d}x}A(x,-x)+$$
$$+\int_{-x}^{+x}\left\{\frac{\partial^2 A}{\partial t^2}-\frac{\partial^2 A}{\partial x^2}-V(x)A\right\}e^{kt}\,\mathrm{d}t=0\,. \tag{4.11}$$

Since $A(x, t)$ shall be independent of $k$, each of the three terms in (4.11) vanishes separately. The second initial condition of (4.9) yields

$$A(0,0)=\frac{1}{2}u'(0)\,.$$

Hence, $A$ is subject to the conditions

$$\frac{\partial^2 A}{\partial x^2}-\frac{\partial^2 A}{\partial t^2}=V(x)A\,, \tag{4.12}$$

$$A(x,x)=\frac{1}{2}\left\{u'(0)+\int_0^x V(t)\,\mathrm{d}t\right\}, \tag{4.13}$$

$$A(x,-x)=\frac{1}{2}u'(0)\,, \tag{4.14}$$

which determine $A$ uniquely, if $u(x)$ is given, since the solution of the hyperbolic equation (4.12), whose characteristics are the lines $x\pm t=$const., is completely specified by its values on a pair of intersecting characteristics [here $x-t=0$ (4.13) and $x+t=0$ (4.14)]. If $f_-$ instead of $f_+$ is used, the same conditions are obtained. Hence, the kernels for $f_+$ and $f_-$ are identical. $A(x, t)$ vanishes for $|t|>x$; its domain of definition is illustrated in Fig. 3.

The kernel $A$ is the link between the data $c(k)$ and the unknown function $\sigma(z)$. The relation between $A$ and $\sigma$ is quite simple. First it is seen from (4.5), (4.6), (4.4), and (4.9) that in the limit $k\to 0$ the functions $u$ and $f_\pm$ satisfy identical differential equations and initial conditions. Hence,

$$u(x)=\lim_{k\to 0}f_\pm(x,k)=1+\int_{-x}^{+x}A(x,t)\,\mathrm{d}t\,. \tag{4.15}$$

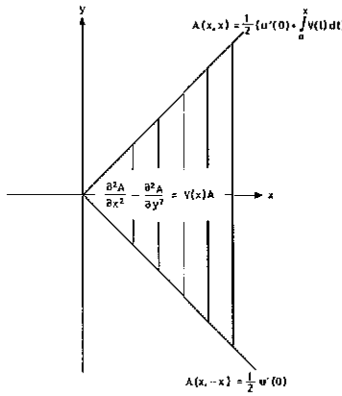



Fig. 3: Definition of the function $A(x, y)$.

Then in virtue of (4.2) and (4.4) the expression of $\sigma(z)$ in terms of $u(x)$ leads to the parameter representation

$$\sigma(z)=\sigma(0)\,u^4(x)\,, \tag{4.16}$$

$$z=\int_0^x u^{-2}(x)\,\mathrm{d}x\,. \tag{4.17}$$

A second solution of (4.6) is $g(x)=u(x)\cdot z(x)$ with $g(0)=0$ and $g'(0)=1$. Hence, the depth $z$ is alternatively determined by

$$z=\lim_{k\to 0}\frac{f_+(x,k)-f_-(x,k)}{2\,ku(x)}=\left\{x+\int_{-x}^{+x}A(x,t)\,t\,\mathrm{d}t\right\}\Big/u(x)\,. \tag{4.17a}$$

Eq. (4.17a) has the advantage that only $u(x)$ is needed instead of all values of $u$ with arguments less than $x$, as in (4.17).

The relation between $A(x, t)$ and $c(k)$ is more complicated and leads to an integral equation. The representation of $f$ by $f_+$ and $f_-$ yields

$$kc(k)\,f(x,k)=f_-(x,k)-b(k)\,\{f_+(x,k)+f_-(x,k)\}\,, \tag{4.18}$$

where

$$b(k)=\frac{1}{2}\,\{1-kc(k)\}\,, \tag{4.19}$$

and the initial conditions (4.9) and the result

$$f'(0,k)=E'(0,\omega)/E(0,\omega)+u'(0)=-1/c(k)+u'(0)$$

have been applied. Insertion of (4.10) into (4.18) leads to

$$kc(k)f(x,k)-e^{-kx}=\int_{-x}^{x} A(x,t)\,e^{-kt}\,dt-b(k)(e^{kx}+e^{-kx})-$$
$$-b(k)\int_{-x}^{x} A(x,t)(e^{kt}+e^{-kt})\,dt\,. \tag{4.20}$$

Now multiply (4.20) by $e^{ky}/(2\pi i)$, $|y|<x$, and integrate over $k$ along the line $k=\varepsilon$, $\varepsilon>0$. The result is abbreviated as

$$I_1=I_2+I_3+I_4\,, \tag{4.21}$$

where $I_1$ to $I_4$ denote the integrals resulting from the four terms of (4.20). Their values are determined as follows.

$I_1$: For $k\to\infty$ the asymptotic representation of $f$ is

$$f(x,k)=\exp\left\{-kx+\frac{1}{2k}\int_0^x V(t)\,dt+O(k^{-2})\right\},$$

provided that $\sigma(z)$ is continuous [Kamke 1959, p. 138]. Hence, the integrand behaves like $\exp\{-k(x-y)\}$, and because of $|y|<x$ the contour can be closed by a large semicircle in the right half-plane without affecting the value of the integral. In the interior the integrand is analytic. Hence,

$$I_1=0\,.$$

$I_2$: The two-sided Laplace transform

$$g(k)=\int_{-\infty}^{+\infty} G(t)\,e^{-kt}\,dt\,,\quad G(y)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} g(k)\,e^{ky}\,dk \tag{4.22}$$

yields immediately

$$I_2=A(x,y)\,.$$

$I_3$: In the sequel the "frequency" function $b(k)$, which is computed from the data, is replaced by the "position" function

$$B(x)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} b(k)\,e^{kx}\,\mathrm{d}k. \tag{4.23}$$

Eqs. (4.19) and (4.8) imply $b(k^*)=b^*(k)$. Hence, $B(x)$ is real. Moreover,

$$B(x)=0 \text{ for } x<0, \tag{4.24}$$

since in this case the contour can be closed by a large semicircle in the right half of the $k$-plane, and the integral is analytic in its interior. — From (4.19), (2.21), (4.4), and (4.2) follows

$$b(k)=\frac{1}{2}u'(0)\,k^{-1}+\mathrm{O}(k^{-2}) \tag{4.25}$$

for $k\to\infty$. Hence,

$$B(+0)=\frac{1}{2}u'(0).$$

Consequently, $B$ is in general discontinuous across $x=0$. The calculation of $B(x)$ turns out to be the crucial step in practical applications. Since experimental data are known only on the bisectors $k=|k|\exp(\pm i\pi/4)$ and a deformation of the contour in (4.23) in direction to the bisectors is not possible, Eq. (4.23) involves analytic continuation of the data in direction to the singularities on the imaginary $k$-axis and in the left half-plane, which is an unstable process. Practicable methods are discussed in Sec. 5. — The provisional result is

$$I_3=-B(x+y),$$

since the term containing $e^{-kx}$ would give $-B(-x+y)$, which vanishes in virtue of $|y|<x$ and (4.24).

$I_4$: The convolution theorem for the two-sided Laplace transform

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} g_1(k)\,g_2(k)\,e^{ky}\,\mathrm{d}k=\int_{-\infty}^{+\infty} G_1(t)\,G_2(y-t)\,\mathrm{d}t$$

(using the notation of (4.22)) yields at once

$$I_4 = -\int_{-x}^{+x} A(x,t)\{B(y+t)+B(y-t)\}\,\mathrm{d}t,$$

since $A(x,t)=0$ for $|t|<x$.

Hence, (4.21) reads explicitly

$$A(x,y)=B(x+y)+\int_{-x}^{+x} A(x,t)\{B(y+t)+B(y-t)\}\,\mathrm{d}t, \quad |y|<x, \tag{4.26}$$

which is a linear integral equation for $A(x,y)$. The variable $x$ enters as a parameter only, the proper variables are $y$ and $t$. Eq. (4.26) has to be solved for all $x$, e.g. by decomposition into a linear system on using Gauss' integration method. When $A$ is found, the conductivity profile is obtained from (4.15)–(4.17a).

Since the solution of the inverse geomagnetic induction problem is known to be unique [Tichonov 1965, Bailey 1970], the uniqueness of a solution of (4.26) will not be proved here.

Finally it should be mentioned that an alternative integral equation could have been obtained by introducing in (4.18) instead of $b(k)$ the function

$$r(k)=\frac{1-kc(k)}{1+kc(k)} \text{ with } R(x)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} r(k)\,e^{kx}\,\mathrm{d}k, \quad \varepsilon>0, \tag{4.27}$$

leading to

$$A(x,y)=R(x+y)+\int_{-y}^{x} A(x,t)\,R(y+t)\,\mathrm{d}t, \tag{4.28}$$

which is formally simpler than (4.26). However, the formulation in terms of $B(x)$ is preferred here, since the determination of $B(x)$ has computational advantages, as will become apparent in the next section. Moreover, numerical experiments have shown that, given exact values of $R(x)$ and $B(x)$, the results obtained from (4.26) were slightly better than those from (4.28).

So far no physical meaning can be attributed to the somewhat abstract functions $A$, $B$, and $R$. Only if the physical situation is changed a simple interpretation of $B$ and $R$ is possible. Consider a non-absorbing elastic medium with wave velocity

$$v(z)=v_0\cdot\sqrt{\sigma(0)/\sigma(z)}$$

in $z\geq 0$, where $v_0$ is arbitrary, and assume that a unit $\delta$-impulse is released at time $t=0$ at $z=+0$, propagating downwards. Then the reflected amplitude recorded at $z=0$ between $t$ and $t+\mathrm{d}t$ is $v_0R(v_0t)\,\mathrm{d}t$ or $v_0B(v_0t)\,\mathrm{d}t$, according whether the wave velocity in $z < 0$ is $v_0$ or infinite. In the former case there is no reflection at $z=0$, whereas in the latter case the impulse is multiply reflected at the surface (reflection coefficient $-1$). This is illustrated by the discontinuity model $\sigma(z)=\sigma_0$, $0 \leq z < d$, $\sigma(z)=\sigma_1$, $z > d$ (which, however, is not tractable by the present inversion procedure, cf. [WEIDELT 1970], p. 66). Then

$$R(x)=r_0\,\delta(x-2d), \quad B(x)=-\sum_{n=1}^{\infty}(-r_0)^n\,\delta(x-2nd),$$

where

$$r_0=\frac{1-\sqrt{\sigma_0/\sigma_1}}{1+\sqrt{\sigma_0/\sigma_1}}$$

is the reflection coefficient at $z=d$.

## 5. Computation of $B(x)$

When $B(x)$ is known the solution of the integral equation (4.26) presents almost no numerical difficulties. The really difficult step in the solution of the inverse problem is the computation of $B(x)$ from the data $b(k)$. Two practicable methods are described in this section; neither, however, turns out to be completely satisfactory.

a) The inversion of (4.23) yields

$$b(k)=\int_0^{\infty} B(x)\,e^{-kx}\,\mathrm{d}x\,, \tag{5.1}$$

i.e. a Laplace integral equation, which can be solved using the values of $b(k)$ on the line $k=|k|\,e^{i\pi/4}$ (cf. [TITCHMARSH 1948], p. 316). Let

$$s=\sigma+i\tau$$

be a new complex variable, multiply (5.1) by $k^{-s}/\Gamma(1-s)$, and integrate along the line $k=|k|\,e^{i\pi/4}$. The resulting left-hand integral

$$M(s)=\frac{1}{\Gamma(1-s)}\int_0^{\infty e^{i\pi/4}} b(k)\,k^{-s}\,\mathrm{d}k \tag{5.2}$$

exists in $0 < \sigma < 1$, since the integrand is $O(k^{-\sigma})$ for $k\to 0$ and $O(k^{-\sigma-1})$ for $k\to\infty$, a consequence of (4.25). In the resulting right-hand integral the order of integration can be changed. Hence, on using the result

$$\int_0^{\infty e^{i\pi/4}} k^{-s} e^{-kx}\, dk = x^{s-1}\,\Gamma(1-s)$$

and the Mellin transform

$$g(s) = \int_0^{\infty} G(x)\, x^{s-1}\, dx\,, \quad G(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} g(s)\, x^{-s}\, ds\,,$$

the solution of (5.1) is

$$B(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)\, x^{-s}\, ds\,, \tag{5.3}$$

where $\sigma = 1/2$ is taken, although any value in $0 < \sigma < 1$ is permitted. Deforming the contour in (5.2) to the positive real axis it is seen that $M(s^*) = M^*(s)$. Hence, $B(x)$ is real. Eq. (5.2) is suitable for $\tau < 0$, since it leads to a decrease $e^{-\pi|\tau|/4}$ of $k^{-s}$ for $\tau \to -\infty$, whereas $M(\sigma + i\tau)$ for $\tau > 0$ is obtained either from $M(\sigma + i\tau) = M^*(\sigma - i\tau)$ or by rotating the line of integration through $-\pi/2$.

A comment on (5.2) and (5.3) is necessary. In (5.3) $M(s)$ is required for large imaginary argument $\tau$, for which $1/\Gamma(1-s)$ in (5.2) is $O\{\exp(\pi|\tau|/2)\}$. This exponential increase is cancelled *in theory* by the integral in (5.2), which is $O\{\exp(-\pi|\tau|/2)\}$, as becomes evident, when the line of integration in (5.2) is rotated through $+\pi/4$ for $\tau < 0$ and $-3\pi/4$ for $\tau > 0$, using the fact that $b(k)$ is regular for $Re\, k > 0$. *Experimental data*—in particular, if they are not very smooth—will not always lead to a $b(k)$, which is regular for $Re\, k > 0$. Hence, in practical applications the possible exponential increase of $M(s)$ for $|\tau| \to \infty$, which prevents the convergence of (5.3), must be replaced by a suitable decrease. This method is not without bias, but it enables the interpretation of data, which do not correspond to any conductivity profile.

b) An alternative approach takes into account particular properties of the response function. Introduce into (2.19) the new variable

$$\mu = \sqrt{\mu_0 \sigma(0)\, \lambda}$$

and let $g(\mu) = \pi\mu a(\lambda)$. Then (2.19) reads

$$c(k) = \frac{2}{\pi}\int_0^{\infty} \frac{g(\mu)\, d\mu}{\mu^2 + k^2}\,, \quad g(\mu) \geq 0\,. \tag{5.4}$$

For a uniform half-space $g(\mu) \equiv 1$, for any other profile $g(\mu) \to 1$ for $\mu \to \infty$. The expression of $B(x)$ in terms of $g(\mu)$ is

$$B(x) = \frac{1}{\pi} \int_0^\infty \{1 - g(\mu)\} \cos \mu x \, \mathrm{d}\mu \,, \tag{5.5}$$

which is easily verified by solving (5.5) for $g(\mu)$, inserting the result into (5.4), changing the order of integration, and integrating over $\mu$. The resulting equation agrees with (5.1).

When $g(\mu)$ is known, the determination of $B(x)$ from (5.5) presents no difficulty. Hence, the actual problem in the inversion procedure is the solution of the integral equation (5.4). The decomposition of (5.4) into linear equations leads to a system, which is badly ill-conditioned. But much of the non-uniqueness of its solution is removed, when it is taken into account that the unknown function $g(\mu)$ must be real and non-negative. A linear system of equations with linear constraints can be solved by quadratic programming techniques, e. g. by the method of Wolfe [COLLATZ & WETTERLING 1971]. Quadratic programming has been proved useful already in the solution of the inverse problem of geoelectrical sounding [KUNETZ & ROCROI 1970], where in fact the same properties of the spectral function are used to advantage (although in a different context).

## 6. An analytical example

In this section the inversion procedure is summarized by a simple analytical example. The general operations are listed on the left-hand side of Table 1, the corresponding outcome is given on the right-hand side. $J_1$ and $I_1$ denote the ordinary and modified Bessel function of the first order, respectively. In applications the inversion ends up with the parameter representation for $\sigma$ and $z$; in this analytical example after elimination of the parameter $x$ a closed expression for $\sigma(z)$ can be obtained. The dependence of the conductivity profile on the wave number $\varkappa$ of the inducing field is given in the last line. For $\varkappa = 0$ there is a perfect conductor at $z = 1/a$, whereas for the largest wave number $\varkappa = a$ (cf. Eq. (3.7)) the conductivity is uniform. – The function $g(\mu)$ (cf. Eqs. (5.4) and (5.5)) is

$$0 \text{ for } 0 \le \mu \le a \text{ and } \mu/\sqrt{\mu^2 - a^2} \text{ for } \mu > a \,.$$

## 7. Inversion of experimental data

The inversion procedure has been applied to a sounding curve obtained by WIESE [1965] at Ückermünde (53°45′ N, 14°04′ E). The data cover the broad period range from 50 sec to 24 h. The disadvantage of the data is the fact that the station is situated in the region of the EW-striking North German conductivity anomaly leading to a

Table 1

| | |
|---|---|
| $c(k)$ | $=1/\sqrt{a^2+k^2}, \quad a\geq 0$ |
| $b(k)=\frac{1}{2}\{1-kc(k)\}$ | $=\frac{1}{2}\{1-k/\sqrt{a^2+k^2}\}$ |
| $M(s)=\frac{1}{\Gamma(1-s)}\int\limits_0^{\infty e^{i\pi/4}} b(k)\,k^{-s}\,\mathrm{d}k$ | $=\frac{1}{2}\left(\frac{a}{2}\right)^{1-s}\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{3-s}{2}\right)}$ |
| $Re\,s=\frac{1}{2}; \quad Im\,s<0; \quad M(s)=M^*(s^*) \quad \text{for} \quad Im\,s>0$ | |
| $B(x)=\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)\,x^{-s}\,\mathrm{d}s$ | $=\frac{a}{2}J_1(ax)$ |
| $x>0; \quad B(x)=0 \quad \text{for} \quad x<0$ | |
| Solve the integral equation (4.26) $A(x,y)=B(x+y)+\int\limits_{-x}^{+x} A(x,t)\{B(y+t)+B(y-t)\}\,\mathrm{d}t, \quad \lvert y\rvert<x:$ | |
| $A(x,y)$ | $=\frac{a}{2}\,\frac{x+y}{\sqrt{x^2-y^2}}I_1(a\sqrt{x^2-y^2})$ |
| $u(x)=1+\int\limits_{-x}^{+x} A(x,t)\,\mathrm{d}t$ | $=\cosh ax$ |
| $\sigma(z)/\sigma(0)=u^4(x)$ | $=\cosh^4 ax$ |
| $z=\begin{cases}\int\limits_0^x u^{-2}(t)\,\mathrm{d}t\\ \left\{x+\int\limits_{-x}^{+x} A(x,t)\,t\,\mathrm{d}t\right\}/u(x)\end{cases}$ | $=a^{-1}\tanh ax$ |
| $\sigma(z)/\sigma(0)$ | $=(1-a^2z^2)^{-2}$ |
| $\sigma_\varkappa(z)=\cosh^{-4}\varkappa z\cdot\sigma(\varkappa^{-1}\tanh\varkappa z)$ | $=\sigma(0)\{\cosh^2\varkappa z-(a/\varkappa)^2\sinh^2\varkappa z\}^{-2}$ |
| $0\leq\varkappa c(0)\leq 1$ | $0\leq\varkappa/a\leq 1$ |

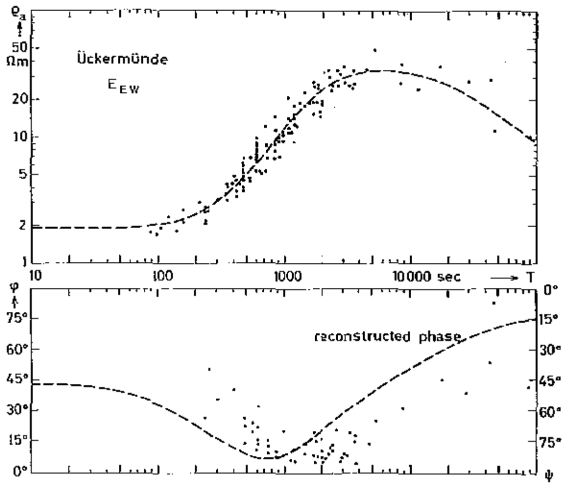



Fig. 4: Input data (top) and reconstruction of phase by (2.27) (bottom).

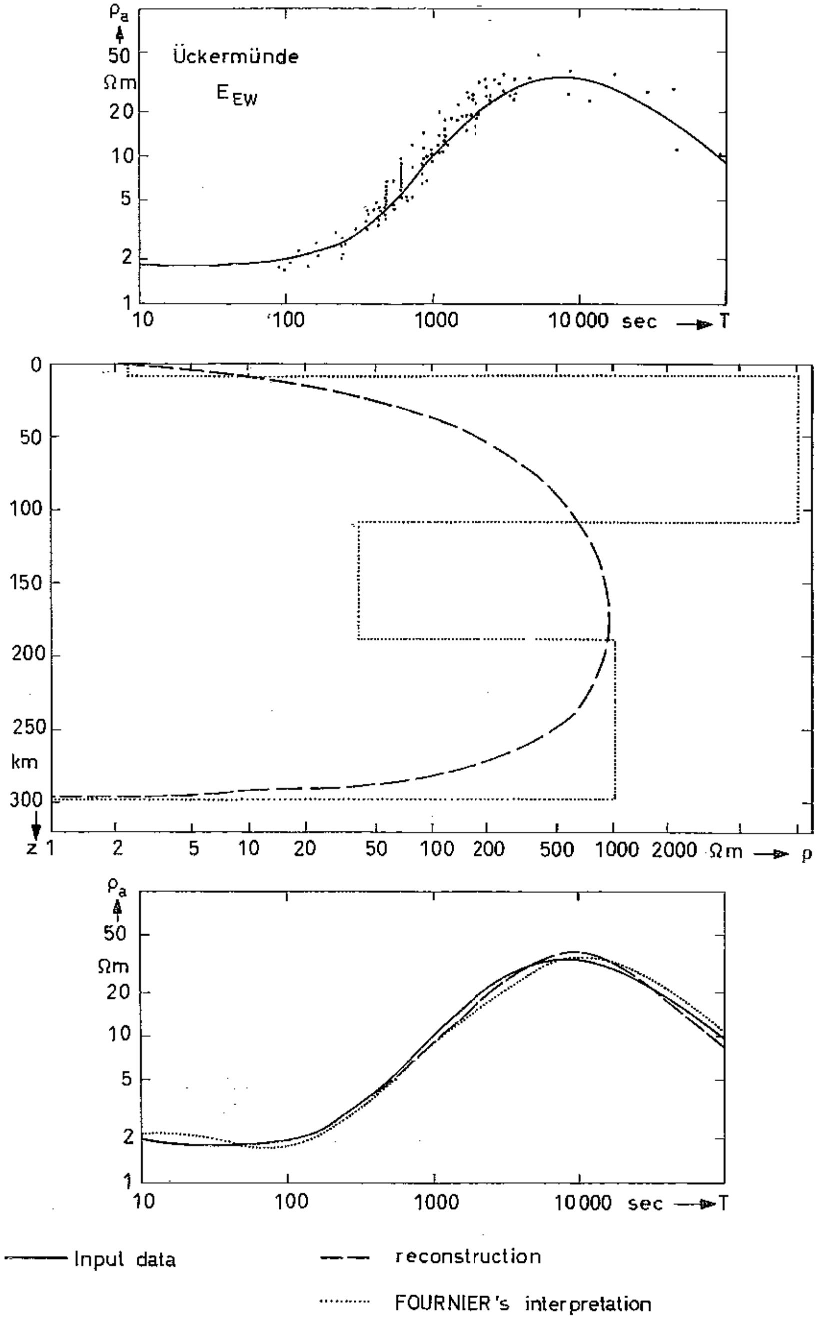
pronounced directivity in the apparent resistivity (cf. Figs. 12 and 13 in the paper of WIESE). Since the electric field component parallel to the strike is less influenced by the anomaly than the perpendicular component, the $\varrho_a$-curves computed from $E_{\mathrm{EW}}$ and $H_{\mathrm{NS}}$ will give the most reliable results when interpreted as the sounding curve of a laterally uniform earth. The data are shown in Fig. 4 (top).

WIESE has also determined the phases, which are compared in Fig. 4 (bottom) with those computed from $\varrho_a$ by (2.27). The phase curves are in qualitative agreement, but there is a systematical phase shift. The reconstructed phase is used for the following inversion. The results of it are shown in Fig. 5 (centre), where they are compared with the results of FOURNIER [1968], who interpreted the same data by a five-layer model postulating a low-resistivity layer in the upper mantle as magnetotelluric evidence for the low-velocity layer of seismic waves. The $\varrho_a$-curves corresponding to the two resistivity profiles are given at the bottom of Fig. 5.

The present example clearly displays the lack of uniqueness of the magnetotelluric method, when a poor conductor, which is electrodynamically little effective, is embedded between two good conductors. To fit the data the resistivity of the poor conductor has to be beyond a certain limit, but can be almost arbitrary otherwise. In the present case the sounding curve essentially fixes only three parameters of the resistivity profile: The horizontal part for short periods specifies the surface resistivity, the 45-degree

---

Fig. 5: Results of the inversion compared with those of FOURNIER.


Ückermünde
$E_{EW}$
$\rho_a$
Ωm
sec → T
km
z
Ωm → ρ
Input data
reconstruction
FOURNIER's interpretation

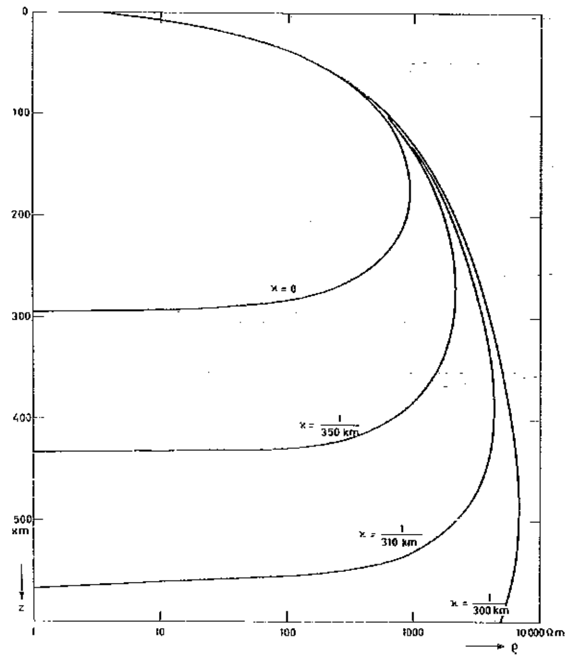



Fig. 6: The dependence of the resistivity profile on the wave number of the external field. (The depth of the perfect conductor for $\varkappa=0$ is slightly less than 300 km.)

ascent for intermediate periods determines the integrated conductivity $\tau$ of the surface layers. Here, approximately,

$$\log \varrho_a(T) = \log T - \log(2\pi\mu_0\tau^2). \tag{7.1 a}$$

The 45-degree descent for long periods stipulates the depth $z_m$ of a perfect conductor:

$$\log \varrho_a(T) = -\log T + \log(2\pi\mu_0 z_m^2), \quad T \to \infty. \tag{7.1 b}$$

The integrated conductivity as determined from the $\varrho_a$-curve is $\tau = 3.5 \cdot 10^3\ \Omega^{-1}$, in agreement with FOURNIER's result and close to $\tau = 3.8 \cdot 10^3\ \Omega^{-1}$ of the continuous model (integrated as far as the resistivity maximum).

Finally the dependence of the resistivity profile on the wave number $\varkappa$ (cf. Sec. 3) is illustrated in Fig. 6. There is an appreciable influence only, if $1/\varkappa$ is slightly greater than $z_m \approx 300$ km, corresponding to a wave length of approximately 2000 km. The natural inducing fields probably have a much greater wave length [SCHMUCKER 1970, p. 92].

## 8. Conclusion

The procedure given in this paper is a practicable way to solve the inverse problem of geomagnetic induction. The results of the previous section, however, cast serious doubts on the usefulness of this method. It just results a smooth resistivity curve compatible with the data, which may be far from other feasible resistivity profiles postulated for physical reasons. Besides, the procedure is rather awkward and needs precise data over a broad frequency range. Hence, it appears that the best way to handle geomagnetic induction data is still to interpret them by a set of homogeneous layers and to introduce, if necessary, further preconceived model assumptions. The merit of the proposed method, however, is that it offers some insight into the nature of the inverse problem.

The real problem in the inversion of geomagnetic induction data is not the method of obtaining a feasible resistivity profile, but the method of finding a reliable estimate of the accuracy and resolving power of the results when the errors of the data are taken into account. A first step in this direction has been done recently by Parker [1970].

## Acknowledgement

I am grateful to Prof. Dr. M. Siebert for stimulating and supporting the research laid down in this paper, which is essentially based on a thesis prepared at the Institut für Geophysik at Göttingen.

## Appendix A

### Proofs of the inequalities given in Sec. 2

For simplicity $\int \sqrt{\sigma}\, dz$ is assumed to be finite. Then according to (2.18) $c(\omega)$ can be represented as

$$c(\omega) = \sum_{n=1}^{\infty} \frac{a_n}{b_n + i\omega}, \quad a_n > 0, \; b_n \geq 0. \tag{A1}$$

The following results, however, apply to the general case (2.19) as well.

Assume $\omega \geq 0$. Then separation of (A1) into real and imaginary parts leads immediately to (2.30a, b). Now let

$$s_k = 2 \sum_{n=1}^{\infty} \frac{a_n b_n^{3-k} \omega^k}{(b_n^2 + \omega^2)^2}, \quad k = 0, 1, 2, 3, \tag{A2}$$

where all $s_k$ are non-negative. Then with the operator D, defined by (2.29),

$$\begin{aligned} s_0 &= 2g + \mathrm{D}g, & s_1 &= h + \mathrm{D}h, \\ s_2 &= -\mathrm{D}g, & s_3 &= h - \mathrm{D}h. \end{aligned} \tag{A3}$$

The inequality of Schwarz yields

$$s_1^2 \le s_0 s_2 , \qquad s_2^2 \le s_1 s_3 , \tag{A4}$$

from which (2.33) is obtained after inserting (A3) and rearranging the terms. Eq. (2.31) follows directly from (A3), and (2.32) is a consequence of (2.5) and

$$\begin{aligned} -|c|\,\mathrm{D}|c| = -\frac{1}{2}\mathrm{D}|c^2| &= s_0 s_2 + 2 s_2^2 + 2 s_3^2 + (s_0 s_2 - s_1^2) = \\ &= |c^2| - s_0^2 - 2 s_1^2 - s_1 s_3 - (s_1 s_3 - s_2^2) . \end{aligned}$$

For inequalities involving derivatives up to the second order let

$$t_k = 8 \sum_{n=1}^{\infty} \frac{a_n b_n^{5-k} \omega^k}{(b_n^2 + \omega^2)^3} , \quad k = 0, \ldots, 5 .$$

Again all $t_k$ are non-negative. It is easily verified that

$$\begin{aligned} t_0 &= 8\,g + 6\,\mathrm{D}g + \mathrm{D}^2 g , & t_1 &= 3\,h + 4\,\mathrm{D}h + \mathrm{D}^2 h , \\ t_2 &= \quad -2\,\mathrm{D}g - \mathrm{D}^2 g , & t_3 &= \ h \qquad - \mathrm{D}^2 h , \\ t_4 &= \quad -2\,\mathrm{D}g + \mathrm{D}^2 g , & t_5 &= 3\,h - 4\,\mathrm{D}h + \mathrm{D}^2 h . \end{aligned} \tag{A5}$$

The following seven inequalities apply:

$$\begin{aligned} & t_0 t_2 - t_1^2 \ge 0 , \\ & t_0 t_4 - t_1 t_3 \ge t_1 t_3 - t_2^2 \ge 0 , \\ & t_0 t_5 - t_1 t_4 \ge t_1 t_4 - t_2 t_3 \,(\ge 0) , \\ & t_1 t_5 - t_2 t_4 \ge t_2 t_4 - t_3^2 \ge 0 , \\ & t_3 t_5 - t_4^2 \ge 0 . \end{aligned} \tag{A6}$$

Four of them are an immediate consequence of Schwarz's inequality, the remaining follow from the fact that $f(k) = t_{2p-k}\, t_k$ is a convex function (i.e. $f'' \ge 0$) implying $f(k+1) - f(k) \ge f(k) - f(k-1)$. Insertion of (A5) into (A6) leads to seven strong but involved inequalities. From these the simple, but rather weak constraints (2.34a, b) are derived by linear combination:

$$\begin{aligned} & 4(h^2 - |\mathrm{D}^2 c|^2) = (t_1 t_3 - t_2^2) + 2(t_2 t_4 - t_3^2) + (t_3 t_5 - t_4^2) \ge 0 , \\ & 4(g^2 - |c + 2\,\mathrm{D}c + \mathrm{D}^2 c|^2) = (t_0 t_2 - t_1^2) + 2(t_1 t_3 - t_2^2) + (t_2 t_4 - t_3^2) \ge 0 . \end{aligned} \tag{A7}$$

In (A4), (A6), and (A7) (or (2.33) and (2.34)) equality holds over the full frequency range if (and only if) the sum (A1) consists of one term only. For real conductors the number of terms is always infinite, but in the degenerate case of the thin sheet approximation of PRICE [1949] one single term occurs for the model consisting of a thin sheet of integrated conductivity $\tau$ at $z=0$ and a perfect conductor at depth $z=z_m$ yielding

$$c(\omega)=\frac{z_m}{1+i\omega\mu_0\tau z_m}.$$

## Appendix B

### Conductivity transformations

The transformations given in Sec. 3 are special cases of a more general class, which is stated in this appendix. Let $w(z)$ be a solution of (3.3a), i. e.

$$w''(z)=\{\varkappa^2+i\omega\mu_0\sigma(z)\}\,w(z), \tag{B 1}$$

and let $f(z)$ be a solution of

$$f''(z)=\varkappa^2 f(z). \tag{B 2}$$

Then the two types of transformations

| Type I | Type II |
|---|---|
| $\tilde{z} \;=\int\limits_0^z f^{-2}(t)\,\mathrm{d}t$ | $\tilde{z} \;=\int\limits_0^z f^2(t)\,\sigma(t)\,\mathrm{d}t,$ |
| $\tilde{w}(\tilde{z})=w(z)/f(z)$ | $\tilde{w}'(\tilde{z})=w(z)/f(z),$ |
| $\tilde{\sigma}(\tilde{z})=f^4(z)\,\sigma(z)$ | $\tilde{\sigma}(\tilde{z})=f^{-4}(z)\,\sigma^{-1}(z)$ |

reduce (B1) to

$$\tilde{w}''(\tilde{z})=i\omega\mu_0\tilde{\sigma}(\tilde{z})\,\tilde{w}(\tilde{z}), \tag{B 3}$$

where the new response function $\tilde{c}(\omega)$ is given by

$$\tilde{c}(\omega)=\begin{cases}\{f^2(0)/c(\omega)+f(0)f'(0)\}^{-1} & \text{for Type I}\\ \{f^2(0)/c(\omega)+f(0)f'(0)\}/(i\omega\mu_0) & \text{for Type II}.\end{cases}$$

The invariant of all transformations is the differential $\sqrt{\sigma}\,\mathrm{d}z$.

The transformations of Sec. 3, leaving $c(\omega)$ unchanged, belong to Type I with $f(0)=1, f'(0)=0$, i. e. $f(z)=\cosh \varkappa z$. If the constant $\varkappa^2$ in (B1) and (B2) is replaced by any function of $z$, the same formulae apply. Hence, after the appropriate change of the independent variable, the function $f(\varrho)$ used in (3.5b) for a spherical earth is the solution of $f''(\varrho)=\varrho^{-2} n(n+1) f(\varrho)$ with $f(1)=1$, $f'(1)=0$.

The transformations of Type II reverse the conductivity profile replacing well conducting regions by poor conductors, and vice versa. If $f'(0)=0$ and $f(0)=1/\sqrt{\sigma(0)}$, then $k\tilde{c}(k)=\{kc(k)\}^{-1}$, $k=\sqrt{i\omega\mu_0\sigma(0)}$, and the reflection coefficient $r(k)$, Eq. (4.27), only reverses sign. Moreover, the transformations of Type II form the basis for the well-known duality relations of magnetotelluric sounding curves (e. g. [Srivastava] 1967). Also the relations (2.30a, b), (2.32a, b), (2.33a, b), and (7.1a, b) are dual. One relation of each pair can be derived from the other by a transformation of Type II.

## References


Anderssen, R. S.: The character of non-uniqueness in the conductivity modelling problem for the earth. Pure and Appl. Geophys. 80, 238–259, 1970

Bailey, R. C.: Inversion of the geomagnetic induction problem. Proc. Roy. Soc. London 315, 185–194, 1970

Cagniard, L.: Basic theory of the magnetotelluric method. Geophysics 18, 605–635, 1953

Četaev, D. N.: On the solution of the inverse problem in the theory of magnetotelluric prospecting. Bull. (Izv.) Acad. Sci. USSR, Earth Physics, No. 9, 610–611, 1966

Collatz, L., & W. Wetterling: Optimierungsaufgaben (2nd ed.). Heidelberger Taschenbücher 15, Springer, Berlin [etc.] 1971

Eckardt, D. H.: Theory and interpretation of the electromagnetic inpedance of the earth. J. Geophys. Res. 73, 5317–5326, 1968

Faddeev, L. D.: The inverse problem in the quantum theory of scattering. J. Math. Phys. 4, 72–104, 1963

Fournier, H. G.: Proposition d'une méthode pour déterminer la structure du premier millier de kilometres de la terre d'après la résistivité apparante. Acta Geophys. Polonica 16, 215–248, 1968

Gel'fand, I. M., & B. M. Levitan: On the determination of a differential equation from its spectral function [Russ.]. Dokl. Akad. Nauk SSSR 77, 557–560, 1951a

Gel'fand, I. M., & B. M. Levitan: On the determination of a differential equation from its spectral function [Russ.]. Izv. Akad. Nauk SSSR, mat. ser. 15, 309–360, 1951b. Engl. transl. in: Amer. Math. Soc. Transl. Ser. 2, 1, 253–304, 1955

Johnson, I. M., & D. E. Smylie: An inverse theory for the calculation of the electrical conductivity of the lower mantle. Geophys. J. 22, 41–53, 1971

KAMKE, E.: Differentialgleichungen, Vol. 1. Akad. Verlagsges., Leipzig 1959

KUNETZ, G., & J. P. ROCROI: Traitement automatique des sondages électriques. Geophys. Prospect. 18, 157–198, 1970

LAHIRI, B. N., & A. T. PRICE: Electromagnetic induction in non-uniform conductors, and the determination of the conductivity of the earth from terrestrial magnetic variations. Phil. Trans. Roy. Soc. London 237, 509–540, 1939

LANDAU, L. D., & E. M. LIFSCHITZ: Lehrbuch der theoretischen Physik. Akademie Verlag, Berlin 1966/67

MEYER, J.: Die magneto-tellurische Tiefensondierung und ihr erdmagnetisches Analogon. Gerlands Beitr. Geophys. 75, 289–300, 1966

MORSE, P. M., & H. FESHBACH: Methods of theoretical physics, Vol. I, McGraw-Hill, New York [etc.] 1953

PARKER, R. L.: The inverse problem of electrical conductivity in the mantle. Geophys. J. 22, 121–138, 1970

PRICE, A. T.: The induction of electric currents in non-uniform thin sheets and shells. Quart. J. Mech. Appl. Math. 2, 283–310, 1949

PRICE, A. T.: Electromagnetic induction in a semi-infinite conductor with a plane boundary. Quart. J. Mech. Appl. Math. 3, 385–410, 1950

SCHMUCKER, U.: Anomalies of geomagnetic variations in the southwestern United States. Bull. Scripps Inst. Ocean. Univ. Calif. 13, 1970

SIEBERT, M.: Ein Verfahren zur unmittelbaren Bestimmung der vertikalen Leitfähigkeitsverteilung im Rahmen der erdmagnetischen Tiefensondierung. Nachr. Akad. Wiss. Göttingen, math.-phys. Kl., Nr. 2, 1964

SRIVASTAVA, S. P.: Magnetotelluric two- and three-layer master curves. Publ. Dom. Observ. Ottawa 35, 309–316, 1967

STRACHOV, V. N.: Approximate solution of incorrectly posed linear problems in Hilbert space with applications to exploration geophysics. Bull. (Izv.) Acad. Sci. USSR, No. 8, 495–506; No. 9, 581–595, 1969

TICHONOV, A. N.: Mathematical basis of the theory of electromagnetic soundings. USSR Comput. Math. and Math. Phys. 5, 207–211, 1965

TITCHMARSH, E. C.: Introduction to the theory of Fourier integrals (2nd ed.). Oxford University Press, London 1948

TITCHMARSH, E. C.: Eigenfunction expansions associated with second order differential equations (2nd ed.). Oxford University Press, London 1962

WEIDELT, P.: Die Umkehraufgabe der erdmagnetischen Tiefensondierung für horizontal geschichtete Leiter. Diss. Math.-Naturw. Fak. Univ. Göttingen, 1970

WIESE, H.: Geomagnetische Tiefentellurik. Dt. Akad. Wiss. Berlin, Geomagn. Inst. Potsdam, Abh. 36, 1965

YUKUTAKE, T.: Electromagnetic induction in a conductor bounded by an inclined surface. Publ. Dom. Observ. Ottawa 35, 317–353, 1967