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Response characteristics of coincident loop transient electromagnetic systems

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ABSTRACT

The occasional occurrence of persistent sign reversals in coincident loop transient electromagnetic (TEM) measurements stimulates an investigation of possible causes for this effect. By examining the response in the complex frequency plane near the spectrum of freely decaying current modes, it is shown that for any physically reasonable frequency-independent distribution of electrical conductivity and magnetic permeability the voltage response to a step function driving current is of one sign only. Moreover, under the conditions mentioned above, the logarithm of the induced voltage is a decreasing convex function of time. These characteristics are retained for more general time functions of the driving current. The conservation of sign for frequency-independent material parameters supports the assumption of IP effects as a possible mechanism for sign reversals. The latter point is illustrated by a simplified example.

INTRODUCTION

The growing application of transient electromagnetic (TEM) methods in prospecting for ore warrants a more detailed investigation of the signal characteristics under regular conditions. A violation of these properties would indicate that the inductive response to the driving current is contaminated with significant contributions from other sources, such as external noise and induced polarization. This paper presents characteristics of the (formally) simplest TEM system, where the transmitter loop coincides with the receiver loop. This configuration is currently used by the MPP series (Kamenetskij, 1976) and by Sirotem (Buselli and O'Neill, 1977). Although it is generally assumed that the induced voltage is of one sign only, definite sign reversals were reported by Spies (1980), and the role of IP effects as a possible cause of sign reversals was studied by Lee (1975, 1981). Besides such a frequency-dependent conductivity, Spies (1980) cites as possible mechanisms for a sign reversal either magnetic effects caused by a special distribution of magnetic permeability or reflections caused by a particular conductivity distribution. However, Gubatyenko and Tikshayev (1979) showed these mechanisms cannot be relevant for a sign reversal by proving that for any frequency-independent linear medium the induced voltage caused by a unit step excitation is of one sign only.

A different proof of the above result is given herein. It relies on properties of the response in the complex frequency plane near the decay spectrum of freely decaying current modes. Moreover, the results of Gubatyenko and Tikshayev (1979) are extended by considering more general time functions of the driving current and by formulating constraints also for slope and curvature of the decay curve.

Let $I(t)$ and $I'(t)$ be the driving current and its time derivative. Assume $\tilde{I}'(t) = 0$ for $t > 0$ and that for ≤ 0, the Laplace transform of $I'(t)$,

$$\gamma(\lambda) = \int_{-\infty}^{+0} \tilde{I}'(t) e^{\lambda t} dt, \tag{1}$$

is of constant sign for $\lambda > 0$. Then the following two general statements hold for $t > 0$:

(1) No sign reversal in the induced voltage $\tilde{U}(t)$ occurs for any distribution of frequency-independent electrical conductivity and magnetic permeability and for any shape of the loop.

(2) Under the same conditions, $\log|\tilde{U}(t)|$ is a decreasing convex function of time, i.e., it has a negative first derivative and a positive second derivative.

A look into a table of Laplace transforms (Abramowitz and Stegun, 1965, chap. 29) shows that the class of driving functions with no sign reversal of $\gamma(\lambda)$ is rather broad. It comprises all monotonic functions, but also oscillating functions such as $\sin(\alpha t - \phi)$, $\alpha > 0$, $0 \leqq \phi \leqq \pi/2$, and half sinusoids. The above statements are clearly true for the simplified conducting loop model of Grant and West (1965, p. 486, 540), where the conducting matter is replaced by a single loop. The following analysis shows that the conducting loop model can be generalized.

THEORY

Let $\sigma(\mathbf{r})$ and $\mu(\mathbf{r})$ denote the electrical conductivity and magnetic permeability, which may be arbitrary, physically reasonable functions of position $\mathbf{r}$. Starting in the frequency domain (time factor $e^{i\omega t}$), using SI units, and neglecting the displacement current, the electric and magnetic field vectors $\mathbf{E}$ and $\mathbf{H}$ and the source current density $\mathbf{j}_s$ are interrelated by

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H},\ \nabla \times \mathbf{H} = \sigma\mathbf{E} + \mathbf{j}_s, \tag{2a, b}$$

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yielding, after elimination of **H**, the differential equation

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} + i\omega\sigma\mathbf{E} = -i\omega\,\mathbf{j}_s. \tag{3}$$

The source current density $\mathbf{j}_s$ can be represented by

$$\mathbf{j}_s(\mathbf{r}, \omega) = I(\omega)\delta_2(\mathbf{r} - \mathbf{r}_s)\hat{\mathbf{s}}(\mathbf{r}_s), \tag{4}$$

where $I(\omega)$ is the Fourier transform of the driving current, $\mathbf{r}_s$ is some point of the loop $\mathscr{L}$, $\hat{\mathbf{s}}$ is the unit vector tangential to the loop at $\mathbf{r}_s$, and δ_2 is a two-dimensional δ-function, nonvanishing over the cross-section of the wire forming the loop. The voltage U induced in the loop $\mathscr{L}$ is given by

$$U(\omega) = -\oint_{\mathscr{L}} \mathbf{E}(\mathbf{r}, \omega) \cdot d\mathbf{r} \tag{5}$$

and conveniently written as

$$U(\omega) = i\omega I(\omega) L(\omega), \tag{6}$$

with

$$L(\omega) = a_\infty + g(\omega), \tag{7}$$

where $L(\omega)$ can be considered as the inductance of the system loop-conductor. The constant a_∞ is the limit of $L(\omega)$ for $\omega \rightarrow \infty$. It vanishes if the magnetic flux through the loop is completely coupled with the conductor, i.e., if the entire loop is lying inside the conductor or on its boundary. The definition of a_∞ implies $g(\omega) \rightarrow 0$ for $\omega \rightarrow \infty$. The ordinary free-space self-inductance of the loop is $L(0) \equiv L_0$, which is finite only if the finite thickness of the wire forming the loop is taken into account (e.g., Smythe, 1968, p. 337).

A simple integral property of the response can be deduced from equation (6). Using the Fourier representation

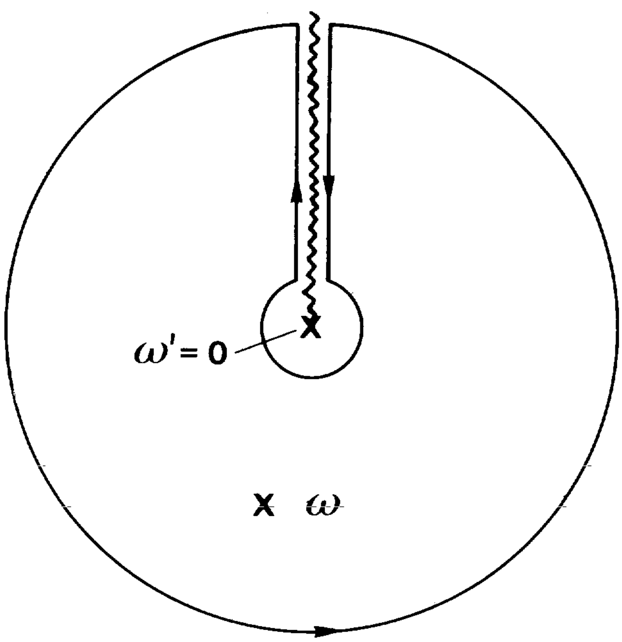

FIG. 1. The contour C in the complex frequency plane.

$$\tilde{F}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega$$

of a time function $\tilde{F}(t)$, the convolution theorem yields

$$\tilde{U}(t) = \int_{-\infty}^{+\infty} \tilde{I}'(\tau) \tilde{L}(t - \tau) d\tau. \tag{8}$$

On integrating over t, we obtain

$$\int_{-\infty}^{+\infty} \tilde{U}(t)\, dt$$

$$= \int_{-\infty}^{+\infty} \tilde{I}'(\tau) d\tau \int_{-\infty}^{+\infty} \tilde{L}(t - \tau) dt = \int_{-\infty}^{+\infty} \tilde{I}'(\tau) d\tau \cdot L(0)$$

or

$$\int_{-\infty}^{+\infty} \tilde{U}(t) dt = L_0\{\tilde{I}(+\infty) - \tilde{I}(-\infty)\}. \tag{9}$$

The right-hand side is independent of the ground. For a unit step driving current, relation (9) was given by Gubatyenko and Tikshayev (1979). It also holds for frequency-dependent material parameters. An experimental verification of integral relation (9) for a step function driving current will be complicated because of the difficulties in recording the dominant contribution of $\tilde{U}(t)$ immediately after the step.

The transfer function

$$\mathbf{e}(\mathbf{r}, \omega) = \mathbf{E}(\mathbf{r}, \omega)/[i\omega I(\omega)], \tag{10}$$

which is physically defined only for real frequencies, can be continued to complex ω. As is well-known from the theory of linear differential equations, a unique solution $\mathbf{e}(\mathbf{r}, \omega)$ of the inhomogeneous equation (3) exists for all ω, for which the associated homogeneous equation

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{f} - \lambda\sigma\mathbf{f} = 0, \quad \omega = i\lambda \tag{11}$$

only has the trivial solution $\mathbf{f} = 0$. At such a point, $\mathbf{e}(\mathbf{r}, \omega)$ is an analytic function of ω (cf., Appendix A). The analyticity breaks down for those ω for which equation (11) has a nontrivial solution subject to the boundary condition of bounded $|\mathbf{f}(\mathbf{r})|$ at infinity. It is easily shown that these free decay modes occur at $\omega = i\lambda$, $\lambda \geqq 0$ (i.e., time factor $e^{-\lambda t}$). After multiplying equation (11) by the complex conjugate solution $\mathbf{f}^*$ and integrating over a sphere S_R of radius R using integration by parts, we obtain

$$\lambda = \lim_{R\rightarrow\infty} \left[\int_{S_R} \frac{1}{\mu} |\nabla \times \mathbf{f}|^2 dv - \oint_{\partial S_R} \frac{1}{\mu} (\mathbf{f}^* \times \nabla \times \mathbf{f}) \cdot d\mathbf{a} \right] \Big/ \int_{S_R} \sigma |\mathbf{f}|^2 dv,$$

where dv is the volume element and $d\mathbf{a}$ the surface element (directed outwards). The behavior of $\mathbf{f}$ for $R \rightarrow \infty$ depends upon σ. If σ is finite at infinity, $\mathbf{f}$ oscillates there and the second quotient is $O(1/R)$. If σ vanishes sufficiently fast, then $\mathbf{f} \rightarrow 0$ for $R \rightarrow \infty$. (This behavior is illustrated in Appendix C by the free decay modes of a uniform half-space.) From both cases, $\lambda \geqq 0$.

The analytic properties of $\mathbf{e}$ are transferred via equations (5)–(7) to $g(\omega)$, implying that this function is analytic outside the positive imaginary axis. Cauchy's integral theorem yields

$$g(\omega) = \frac{1}{2\pi i} \oint_C \frac{g(\omega')}{\omega' - \omega} d\omega',$$

where the closed contour C is shown in Figure 1. Since $g(\omega) \to 0$ for $\omega \to \infty$, the large circle does not contribute to the integral, if its radius approaches infinity. Also the small circle around $\omega' = 0$ yields no contribution in the limit of a vanishing radius, since $g(\omega)$ has to be $O(\omega^{-1+\varepsilon})$, $\varepsilon > 0$, for $\omega \to 0$ to ensure [with reference to equations (6) and (7)] that the induced voltage vanishes in the limit $t \to \infty$ for a step-function driving current. Finally, the Fourier transform of $U(\omega)$ has to be real, which requires that g attains complex conjugate values at points symmetric to the imaginary axis, i.e., $g(\omega) = g^*(-\omega^*)$. Hence,

$$g(\omega) = \int_0^\infty \frac{a(\lambda)\,d\lambda}{\lambda + i\omega}, \tag{12}$$

with

$$a(\lambda) = -\lim_{\varepsilon \to +0} \frac{1}{\pi} \operatorname{Im}[g(i\lambda + \varepsilon)]. \tag{13}$$

The function $a(\lambda)$ is deduced from the differential equation (3) by multiplying it by its complex conjugate solution $\mathbf{E}^*$ and integrating the result over the full space V. After integration by parts, the surface integral vanishes by virtue of the localized source. On using equations (4) and (5), the result is

$$i\omega I U^* = \int_V \left\{\frac{1}{\mu}|\nabla \times \mathbf{E}|^2 + i\omega\sigma|\mathbf{E}|^2\right\} dv.$$

Taking the complex conjugate, dividing by $|\omega I|^2$, and using equation (6), we obtain

$$L(\omega) = \int_V \left\{\frac{1}{\mu}|\nabla \times \mathbf{e}|^2 - i\omega^*\sigma|\mathbf{e}|^2\right\} dv, \tag{14}$$

which implies, from equations (7), (12), and (13),

$$a(\lambda) = \lim_{\varepsilon \to +0} \frac{\varepsilon}{\pi} \int_V \sigma\,|\mathbf{e}|^2 dv \geqq 0. \tag{15}$$

This nonnegativity of $a(\lambda)$ causes the simple characteristics of the coincident loop response. The real part of $L(\omega)$ [equation (14)] is nonnegative for all real frequencies. Consequently, $a_\infty \geqq 0$. Hence, $L(\omega)$ admits the representation

$$L(\omega) = a_\infty + \int_0^\infty \frac{a(\lambda)}{\lambda + i\omega}\,d\lambda,\ a_\infty \geqq 0,\ a(\lambda) \geqq 0. \tag{16}$$

This is illustrated in Appendix B by a simple example. The Fourier transform of equation (16) is

$$\tilde{L}(t) = \begin{cases} 0, & t < 0 \\ a_\infty \delta(t) + \displaystyle\int_0^\infty a(\lambda)e^{-\lambda t}d\lambda, & t \geqq 0, \end{cases} \tag{17}$$

which yields for an arbitrary driving current $\tilde{I}(t)$ the response given in equation (8) (with $t + 0$ as upper integration limit). In particular, the response to a step-function driving current

$$\tilde{I}(t) = \begin{cases} 0, & t < 0 \\ I_0, & t > 0 \end{cases}$$

is $\tilde{U}(t) = I_0\tilde{L}(t)$. If $\tilde{I}'(t) = 0$ for $t > 0$ and $\gamma(\lambda)$ is the Laplace transform of $\tilde{I}'(t)$ for $t \leqq 0$ [cf., equation (1)], then for $t > 0$

$$\tilde{U}(t) = \int_0^\infty \gamma(\lambda)a(\lambda)e^{-\lambda t}d\lambda. \tag{18}$$

A sufficient condition for a constant sign of $\tilde{U}(t)$ for $t > 0$ is a constant sign of $\gamma(\lambda)$ for $\lambda > 0$. Assuming $\gamma(\lambda) \geqq 0$ for ease of notation, we obtain

$$\tilde{U}(t) \geqq 0,\ \tilde{U}'(t) \leqq 0,\ \tilde{U}(t)\cdot\tilde{U}''(t) - [\tilde{U}'(t)]^2 \geqq 0, \tag{19}$$

where the last result was deduced by Schwarz's inequality

$$\int_0^\infty f_1^2 d\lambda \cdot \int_0^\infty f_2^2 d\lambda - \left(\int_0^\infty f_1 f_2 d\lambda\right)^2 \geqq 0$$

with $f_1^2 = \gamma(\lambda)a(\lambda)e^{-\lambda t}$ and $f_2 = \lambda f_1$. Hence, for finite t the voltage $\tilde{U}(t)$ is positive and $\log \tilde{U}(t)$ is a decreasing convex function. It approaches a straight line only when the decay process is dominated by a single decay constant. Equation (19) includes the weaker result that $\tilde{U}(t)$ is also decreasing and convex. The generally used double logarithmic plot of $\tilde{U}(t)$ is decreasing, but no definite sign of the second derivative can be inferred from equation (18).

The same results can be obtained by a free decay mode expansion of $\mathbf{E}$. The arguments are briefly discussed in Appendix C.

IP EFFECTS AND CONCLUSION

The preceding results show that frequency independent conductivities and permeabilities yield coincident loop responses of one sign only, leaving induced polarization (IP) effects as the most probable candidate for persistent sign reversals. This possibility will be demonstrated by an extremely simplified example. In the conducting loop model of Grant and West (1965, p. 486, 540) an extended conductor is replaced by a conducting loop

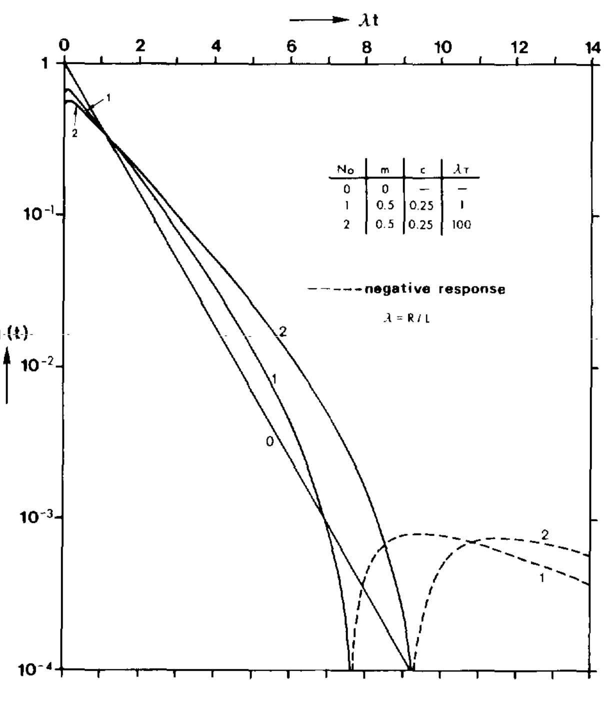

FIG. 2. Semi-logarithmic plots of the normalized voltage $\bar{u} = \tilde{U}/(k^2\lambda I_0 L_0)$ for pure inductive decay (0) and for the decay modified by IP effects (1 and 2).

with self-inductance L and resistance R. If M is the mutual inductance between this loop and the conducting loop, the induced voltage in the frequency domain is

$$U(\omega) = i\omega I(\omega) L_0[(1 - k^2) + k^2 R/(R + i\omega L)], \qquad (20)$$

where $k^2 = M^2/(LL_0) \leqq 1$ is the squared coupling coefficient. It yields the step current response

$$\tilde{U}(t) = I_0 L_0[(1 - k^2)\delta(t) + k^2 \lambda e^{-\lambda t}],\ \lambda = R/L.$$

Pelton et al (1978) succeeded in fitting the frequency dependent impedance $Z(\omega)$ of many different IP spectra by a simple Cole-Cole relaxation model

$$Z(\omega) = R\left[1 - m\left(1 - \frac{1}{1 + (i\omega\tau)^c}\right)\right], \qquad (21)$$

where typically $0.1 \leqq m \leqq 0.9$, $0.1 \leqq c \leqq 0.6$, and the time constant τ, characterizing the IP decay, varies over a broad range. If R in equation (20) is replaced by $Z(\omega)$, the resulting step-function response is shown in Figure 2 for representative values of m, c, and $\lambda\tau$, where the latter is the ratio of the time constants for pure IP decay and pure inductive decay. Induced polarization affects the inductive decay by reducing the decay rate and creating negative responses at late times, as Lee (1975) also found in a more complex example. Recently, Lee (1981) also produced sign reversals by applying model (21) to a uniform half-space. The concave shape of curves 1 and 2 in Figure 2 shows that these curves cannot result from a pure inductive decay. In order to satisfy relation (9), which also holds for model (21), the negative responses have to be balanced by a reduced decay rate at positive responses.

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APPENDIX A
ANALYTICITY OF $\mathbf{e}(\omega)$

The analyticity of the transfer function $\mathbf{e}(\mathbf{r}, \omega)$ [cf., equation (10)] in the complex frequency plane outside the positive imaginary axis has played a key role in the preceding developments. Although this result can be inferred from the analytic properties of the resolvent of a linear operator (e.g., Kantorovich and Akilov, 1964, p. 512), a more elementary justification of this assertion will be given here. It is sufficient to show that for two frequencies ω and ω' outside the positive imaginaries, the limit

$$\lim_{\omega' \to \omega} \frac{\mathbf{e}(\omega') - \mathbf{e}(\omega)}{\omega' - \omega} \qquad (A-1)$$

exists and is independent of the mode in which $\omega' \to \omega$.

Let the subscript $j = 1, 2, 3$ denote the Cartesian directions $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. Then the Green's vector $\mathbf{G}_j(\mathbf{r}_0|\mathbf{r}, \omega)$ associated with equation (3) and vanishing at infinity satisfies

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{G}_j(\mathbf{r}_0|\mathbf{r}, \omega) + i\omega\sigma(\mathbf{r})\mathbf{G}_j(\mathbf{r}_0|\mathbf{r}, \omega) = \hat{\mathbf{x}}_j \delta(\mathbf{r} - \mathbf{r}_0). \qquad (A-2)$$

$\mathbf{G}_j$ is essentially the electric field of an oscillating electric dipole placed in the $\hat{\mathbf{x}}_j$-direction at $\mathbf{r}_0$. According to equations (10), (3), and (4), $\mathbf{e}(\mathbf{r}, \omega)$ is a solution of

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{e}(\mathbf{r}, \omega) + i\omega\sigma(\mathbf{r})\mathbf{e}(\mathbf{r}, \omega) = -\delta_2(\mathbf{r} - \mathbf{r}_s)\hat{\mathbf{s}}(\mathbf{r}_s). \qquad (A-3)$$

Multiply equation (A-2) by $\mathbf{e}(\mathbf{r}, \omega)$ and (A-3) by $\mathbf{G}_j(\mathbf{r}_0|\mathbf{r}, \omega)$ and integrate the difference over the full space. On integrating by parts, the boundary at infinity yields no contribution because of the localized sources. Hence, the jth component of $\mathbf{e}$ is given by

$$e_j(\mathbf{r}_0, \omega) = -\oint_{\mathcal{S}} \mathbf{G}_j(\mathbf{r}_0|\mathbf{r}, \omega) \cdot d\boldsymbol{\tau}. \qquad (A-4)$$

After replacing ω by ω' in equation (A-3), a similar procedure yields [from equation (A-4)]

$$e_j(\mathbf{r}_0, \omega') - e_j(\mathbf{r}_0, \omega) = -i(\omega'V - \omega) \int_V \sigma(\mathbf{r})\mathbf{G}_j(\mathbf{r}_0|\mathbf{r}, \omega) \cdot \mathbf{e}(\mathbf{r}, \omega')dv. \qquad (A-5)$$

To see that the integral in equation (A-5) exists and is bounded, we assume for definiteness a conducting half-space with $\sigma = 0$ in $z < 0$ and $\sigma > 0$ in $z > 0$, where σ and μ should tend to constant values σ_0 and μ_0 at large distances from some origin. Then the asymptotic behavior of $\mathbf{G}_j$ at infinity is nonuniform: near the air-earth interface $\mathbf{G}_j$ will show the $1/|\mathbf{r} - \mathbf{r}_0|^3$ decay of a dipole field in air. Well below the source point the exponential decay of an oscillating dipole in a full space with parameters σ_0 and μ_0 applies, i.e.,

$$|\mathbf{G}_j(\mathbf{r}_0|\mathbf{r},\omega)| = O\left(\frac{e^{-k_0|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\right), \; k_0 = \sqrt{i\omega\mu_0\sigma_0} \quad \text{(A-6)}$$

(e.g., Morse and Feshbach, 1953, p. 1781). Because of equation (A–4), $\mathbf{e}$ has the same asymptotic properties as $\mathbf{G}_j$. Consequently, the integrand in equation (A–5) decays sufficiently fast to yield a bounded integral. This implies that $\mathbf{e}$ is continuous at ω. Hence, a unique limit (A–1) can be formed from (A–5).

The above arguments break down, if ω is positive imaginary. In this case the exponential term in equation (A–6) is purely oscillatory and the field no longer decays sufficiently fast at infinity.

APPENDIX B
EXAMPLE FOR THE REPRESENTATION OF $U(\omega)$ BY EQUATIONS (6) AND (16)

In this appendix the representation of the induced voltage $U(\omega)$ by equation (6) and (16) is illustrated by a simple well studied example, namely a horizontal circular loop of radius b at height h over a uniform half-space of conductivity σ and vacuum permeability μ_0.

The work of Lee and Lewis (1974) infers that a_∞ and $g(\omega)$ [equation (7)] are given by

$$a_\infty = L_0 - \mu_0\pi b^2 \int_0^\infty J_1^2(\kappa b)e^{-2\kappa h}\,d\kappa, \qquad \text{(B-1)}$$

and

$$g(\omega) = 2\pi\mu_0 b^2 \int_0^\infty J_1^2(\kappa b)e^{-2\kappa h}\,\frac{\kappa\,d\kappa}{\kappa+\beta}, \qquad \text{(B-2)}$$

where $\beta = \sqrt{\kappa^2 + i\omega\mu_0\sigma}$, and J_1 is the Bessel function of first kind and first order.

To begin with, it is noted that $1/(\kappa+\beta)$ as a function of ω is analytic outside the positive imaginary ω-axis (where it has a branch point) and vanishes for $\omega \to \infty$. Hence, equations (12) and (13) are also applicable to this function and yield

$$\frac{\kappa}{\kappa+\beta} = \frac{\sqrt{\lambda_0}}{\pi}\int_{\lambda_0}^\infty \frac{\sqrt{\lambda-\lambda_0}\,d\lambda}{\lambda(\lambda+i\omega)}, \; \lambda_0 = \kappa^2/(\mu_0\sigma). \quad \text{(B-3)}$$

Inserting equation (B–3) into equation (B–2) and changing the order of integration (which includes a change in the integration limits), we identify [with reference to equation (12)]

$$a(\lambda) = \frac{2b^2}{\lambda\sigma}\int_0^{\kappa_0} \kappa\sqrt{\kappa_0^2-\kappa^2}\,e^{-2\kappa h}J_1^2(\kappa b)\,d\kappa, \; \kappa_0 = \sqrt{\lambda\mu_0\sigma},$$

which is positive as required. If $h = 0$, further processing is possible by means of the identities (9.1.14) and (6.2.1) of Abramowitz and Stegun (1965), yielding

$$a(\lambda) = \mu_0 b\sqrt{\pi}\sum_{k=0}^\infty \frac{(-1)^k(2k+2)!(\sqrt{\lambda\mu_0\sigma}\,b/2)^{2k+3}}{k!(k+1)!(k+2)!\Gamma(k+7/2)}.$$

After performing the λ-integration in equation (17), the result of Lee and Lewis (1974) or Raiche and Spies (1981) is recovered.

APPENDIX C
FREE DECAY MODE EXPANSION OF $\mathbf{E}(\mathbf{r},\omega)$

In this appendix it is shown that the representation (16) can also be obtained by expanding $\mathbf{E}(\mathbf{r},\omega)$ into the series of free decay modes, which are the bounded nontrivial solutions of equation (11). These vectorial eigenfunctions form an orthogonal set with $\sigma(\mathbf{r})$ as weighting function. For definiteness, a conducting half-space with $\sigma(\mathbf{r}) = 0$ in $z < 0$ and $\sigma(\mathbf{r}) > 0$ in $z > 0$ is assumed.

The nature of the spectrum of decay constants is immaterial for the sequel. But for example, it is reasonable to assume that the well-known results for three-dimensional scalar equations of type

$$\nabla\cdot\left(\frac{1}{\mu}\,\nabla f\right) + \lambda\sigma f = 0, \qquad \text{(C-1)}$$

or

$$\nabla\cdot\left(\frac{1}{\sigma}\,\nabla g\right) + \lambda\mu g = 0, \qquad \text{(C-2)}$$

apply also to the vector equation (11). [This assumption is supported by the fact that equation (11) reduces to the two-dimensional analogs of equations (C–1) and (C–2) in the case of E- and H-polarization, respectively.] From the asymptotic distribution of eigenfunctions of equations (C–1) and (C–2) for $\lambda \to \infty$ (Courant and Hilbert, 1953, p. 442), it is inferred that the spectrum is discrete, if for bounded σ the integral

$$\int_V (\mu\sigma)^{3/2}\,dv$$

is finite, i.e., if σ decreases faster than R^{-2}, where R is the distance from some origin. If the integral diverges, as in all half-space models with nonzero conductivity, the eigenvalue spectrum is continuous for large λ.

However, for simplicity even in the latter case eigenfunctions are denoted symbolically by a discrete quantum number n, and the electric field vector is expanded as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_\infty(\mathbf{r}) + \sum_n c_n\mathbf{f}_n(\mathbf{r}). \qquad \text{(C-3)}$$

The first term on the right-hand side takes into account the possible incompleteness of the set of eigenfunctions, which would occur if both the point of observation and the current loop (or parts of it) were lying in the air half-space. Details will follow below.

Assuming the normalization

$$\int_V \sigma\,\mathbf{f}_n\cdot\mathbf{f}_{n'}^*\,dv = \delta_{nn'},$$

the expansion coefficients c_n are given by

$$c_n = \int_V \sigma\,\mathbf{E}\cdot\mathbf{f}_n^*\,dv = -\frac{i\omega I}{\lambda_n + i\omega}\oint_{\mathcal{L}} \mathbf{f}_n^*\cdot d\mathbf{r}. \quad \text{(C-4)}$$

This result has been obtained by multiplying the complex conjugate of equation (11) for $\mathbf{f} = \mathbf{f}_n$ by $\mathbf{E}$ and equation (3) by $\mathbf{f}_n^*$ and integrating the difference over the full space V. On using equation (5) the induced voltage is

$$U(\omega) = -\oint_{\mathcal{L}} \mathbf{E}_\infty\cdot d\mathbf{r} + i\omega I\cdot\sum_n \frac{a_n}{\lambda_n + i\omega}, \quad \text{(C-5)}$$

where

$$a_n = \left| \oint_{\mathcal{L}} \mathbf{f}_n \cdot d\mathbf{r} \right|^2 \geq 0. \quad \text{(C-6)}$$

The term $\mathbf{E}_\infty$ in equation (C–3) is due to the occurrence of conductivity as weighting function in the orthonormalization of the eigenfunctions; if the conductivity in $z > 0$ tends to infinity, both the amplitudes of the eigenfunctions and the expansion coefficients c_n tend to zero. In this limit, there will be no contribution from the second term in equation (C–3). On the other hand, there exists an electric field in the insulator $z < 0$ consisting of the primary **E**-field and its (negative) image, mirrored at $z = 0$. $\mathbf{E}_\infty$ is equivalent to the high-frequency limit. Hence, we identify, in view of equations (5)–(7),

$$-\oint_{\mathcal{L}} \mathbf{E}_\infty \cdot d\mathbf{r} = i\omega I a_\infty, \quad \text{(C-7)}$$

and

$$\sum_n \frac{a_n}{\lambda_n + i\omega} = g(\omega). \quad \text{(C-8)}$$

In the case of a nondegenerate spectrum, the representations (C–8) and (12) are identical. If the spectrum is degenerate, representation (C–8) sums the contributions from individual eigenfunctions, whereas the integration in equation (12) extends over the eigenvalues.

The representation (C–8) is illustrated by the example of Appendix B: A horizontal circular loop of radius b at height $z = -h$ over a uniform half-space of conductivity σ in $z > 0$. For this model the free decay modes $\mathbf{f}_{\alpha\mathbf{k}}(\mathbf{r})$ are characterized by a continuous wavenumber vector

$$\mathbf{k} = u\,\hat{\mathbf{x}} + v\,\hat{\mathbf{y}} + w\,\hat{\mathbf{z}}$$

with $-\infty < u < \infty$, $-\infty < v < \infty$, $0 \leqq w < \infty$ and by a quantum number α indicating the polarization: for $\alpha = 1$ the current loops are closed in horizontal planes and for $\alpha = 2$ in vertical planes. Using the abbreviations

$$\boldsymbol{\kappa} = u\,\hat{\mathbf{x}} + v\,\hat{\mathbf{y}},\ \kappa = |\boldsymbol{\kappa}|,\ k = |\mathbf{k}|,$$

the orthogonal eigenfunctions $\mathbf{f}_{\alpha\mathbf{k}}$, normalized by

$$\int_{z>0} \sigma\, \mathbf{f}_{\alpha\mathbf{k}} \cdot \mathbf{f}^*_{\alpha'\mathbf{k}'}\, dv = \delta_{\alpha\alpha'}\,\delta(\mathbf{k} - \mathbf{k}'),$$

can be constructed from the general half-space approach of Weaver (1970):

$$\mathbf{f}_{1\mathbf{k}} = \nabla \times (\hat{\mathbf{z}}\psi_{1\mathbf{k}}),$$
$$\mathbf{f}_{2\mathbf{k}} = \nabla \times \nabla \times (\hat{\mathbf{z}}\psi_{2\mathbf{k}}),\ z \neq 0,$$

where

$$\psi_{1\mathbf{k}}(\mathbf{r}) = N(\mathbf{k})e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \cdot \begin{cases} w\, e^{\kappa z}, & z \leqq 0 \\ (w \cos wz + \kappa \sin wz), & z \geqq 0 \end{cases}$$

$$\psi_{2\mathbf{k}}(\mathbf{r}) = N(\mathbf{k})e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \cdot \begin{cases} (w/\kappa)e^{\kappa z}, & z < 0 \\ \sin wz, & z > 0 \end{cases}$$

and

$$N(\mathbf{k}) = 1/[\kappa k(2\pi^3\sigma)^{1/2}].$$

These eigenfunctions belong to the eigenvalue

$$\lambda_{\alpha\mathbf{k}} = k^2/(\mu_0\sigma) = (\kappa^2 + w^2)/(\mu_0\sigma). \quad \text{(C-9)}$$

In this highly degenerate model, the decay constants λ depend only on the modulus of the wavenumber vector and are independent of the polarization. A layered model removes the degeneracy with respect to w and α.

Now it will be shown that the functions $g(\omega)$ defined in equation (C–8) and (B–2) are identical. First it is noted that for any loop in the air half-space the closed contour integrals in equation (C–4) vanish for the polarization $\alpha = 2$, since $\mathbf{f}_{2\mathbf{k}}(\mathbf{r})$ is a potential field in $z < 0$. In this case the current flow is completely described by the horizontal current mode $\alpha = 1$, which is a well-known result. Using the identity (9.1.21) of Abramowitz and Stegun (1965), it is easily found that

$$a(\mathbf{k}) = \left| \oint_{\mathcal{L}} \mathbf{f}_{1\mathbf{k}} \cdot d\mathbf{r} \right|^2 = \frac{2b^2 w^2}{k^2 \pi\sigma} e^{-2\kappa h} J_1^2(\kappa b).$$

From equation (C–8)

$$g(\omega) = \int_{w>0} \frac{a(\mathbf{k})}{\lambda + i\omega}\, d^3\mathbf{k} = 2\pi \int_0^\infty \kappa\, d\kappa \int_0^\infty dw \frac{a(\mathbf{k})}{\lambda + i\omega},$$

where $\lambda = \lambda_{1\mathbf{k}}$ is given by equation (C–9). After performing the elementary w-integration, the remaining κ-integral agrees with equation (B–2).